I. Conditions for Electroquasistatic Fields

A. Order of Magnitude Estimate [Characteristic Length \( L \), Characteristic time \( \tau \)]

\[
\nabla \cdot \vec{E} = \rho/\varepsilon \Rightarrow \frac{\vec{E}}{L} = \frac{\rho}{\varepsilon} \Rightarrow \vec{E} = \frac{\rho L}{\varepsilon}
\]

\[
\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{\vec{H}}{L} = \frac{\varepsilon L}{\tau} \Rightarrow \vec{H} = \frac{\varepsilon L}{\tau} = \frac{L^2 \rho}{\tau}
\]

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \Rightarrow \frac{\vec{E}_{\text{error}}}{L} = \frac{\mu H}{\tau} = \frac{\mu L^2}{\varepsilon^2} \Rightarrow \vec{E}_{\text{error}} = \frac{\mu L^3}{\tau}
\]

\[
\frac{\vec{E}_{\text{error}}}{\vec{E}} = \frac{\mu L^3}{\tau \varepsilon^2} = \frac{\mu c^2 L^2}{(ct)^2} \Rightarrow \frac{1}{c^2} = \frac{1}{\sqrt{\varepsilon \mu}}
\]

\[
\frac{\vec{E}_{\text{error}}}{\vec{E}} \ll 1 \Rightarrow \frac{L}{ct} \ll 1
\]
B. Estimate of Error introduced by EQS approximation

Figure 3.3.2 Plane parallel electrodes having no resistance, driven at their outer edges by a distribution of sources of EMF.

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\[ \bar{E} = \frac{V}{d} \bar{v} = E_0 \bar{v} \]

\[ \sigma_{su} = \begin{cases} -\varepsilon E_0 & z = d \\ \varepsilon E_0 & z = 0 \end{cases} \]

\[ K_r 2\pi b + \pi b^2 \frac{d\sigma_{su}}{dt} = 0 \Rightarrow K_r = -\frac{b}{2} \frac{d\sigma_{su}}{dt} = -\frac{b}{2} \varepsilon \frac{dE_0}{dt} \]

\[ \oint_C \bar{H} \cdot d\bar{s} = \int_S \frac{\partial}{\partial t} (\varepsilon \bar{E}) \cdot d\bar{a} \Rightarrow H_y 2\pi r = \pi r^2 \varepsilon \frac{dE_0}{dt} \Rightarrow H_y = \frac{r}{2} \varepsilon \frac{dE_0}{dt} \]

\[ \oint_C \bar{E} \cdot d\bar{s} = -\int_S \mu \frac{\partial \bar{H}}{\partial t} \cdot d\bar{a} \]

Figure 3.3.3 Parallel plates of Figure 3.3.2, showing volume containing lower plate and radial surface current density at its periphery.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

Figure 3.3.4 Cross-section of system shown in Figure 3.3.2 showing surface and contour used in evaluating correction \( E \) field.

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\[
\left[ E_z(b) - E_z(r) \right] d = \frac{\mu \varepsilon}{2} \int_r^b r' dr' d^2E_0 \frac{d^2E_0}{dt^2}
\]

\[
= \frac{\mu \varepsilon d}{4} \left( b^2 - r^2 \right) \frac{d^2E_0}{dt^2}
\]

\[
E_z(r) = E_0 + \frac{\varepsilon \mu}{4} \frac{d^2E_0}{dt^2} \left( r^2 - b^2 \right)
\]

If \( E_0(t) = A \cos \omega t \)

\[
\frac{|E_{\text{error}}|}{E_0} = \frac{\varepsilon \mu}{4E_0} \frac{d^2E_0}{dt^2} \left( b^2 - r^2 \right) = \frac{1}{4} \omega^2 \varepsilon \mu \left( b^2 - r^2 \right)
\]

\[
\frac{|E_{\text{error}}|}{E_0} \ll 1 \Rightarrow \frac{\omega^2 \varepsilon \mu b^2}{4} \ll 1
\]

\[
f \lambda = c = \frac{1}{\sqrt{\varepsilon \mu}}
\]

\[
\frac{\omega}{2\pi} = \frac{c}{\lambda} = \frac{2\pi c}{\lambda} = \frac{\omega^2 \varepsilon \mu b^2}{4} = \frac{\pi^2 b^2}{\lambda^2} \ll 1 \Rightarrow b \ll \frac{\lambda}{\pi}
\]

\[
f = 1 \text{ MHz in free space} \Rightarrow \lambda = \frac{3 \times 10^8}{10^6} = 300 \text{ m}
\]

If \( b \ll 100 \text{ m} \) \quad EQS approximation is valid.

II. Conditions for Magnetoquasistatic Fields

(b) MQS system consisting of perfectly conducting loop driven by current source. The radius of the loop and diameter of its cross-section are on the order of \( L \).

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.
\[ \nabla \times \vec{H} = \vec{J} \Rightarrow \frac{\vec{H}}{L} = \vec{J} \Rightarrow \vec{H} = \vec{J} L \]

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \Rightarrow \frac{\vec{E}}{L} = \frac{\mu \vec{H}}{\tau} \Rightarrow \vec{E} = \frac{\mu HL}{\tau} = \frac{\mu J L^2}{\tau} \]

\[ \nabla \times \vec{H}_{\text{error}} = \varepsilon \frac{\partial \vec{E}}{\partial t} \Rightarrow \frac{\vec{H}_{\text{error}}}{L} = \frac{\varepsilon \vec{E}}{\tau} \Rightarrow \vec{H}_{\text{error}} = \frac{\varepsilon \vec{E} L}{\tau^2} = \frac{\varepsilon \mu J L^3}{\tau^2} \]

\[ \frac{H_{\text{error}}}{H} = \frac{\varepsilon \mu J L^3}{\tau^2} \frac{\varepsilon \mu L^2}{\tau^2} = \frac{L^2}{(c\tau)^2} \ll 1 \Rightarrow L \ll c\tau \]

**Figure 3.4.1** Range of characteristic times over which quasistatic approximation is valid. The transit time of an electromagnetic wave is \( \tau_{\text{em}} \) while \( \tau_{\Theta} \) is a time characterizing the dynamics of the quasistatic system.

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\[ \tau_{\text{em}} = \frac{L}{c} = L\sqrt{\varepsilon \mu} \]
III. Boundary Conditions

1. Gauss' Continuity Condition

\[ \oint \varepsilon \cdot d\mathbf{a} = \int \sigma_s dS \Rightarrow \varepsilon_0 (E_{2n} - E_{1n}) dS = \sigma_s dS \]

\[ \varepsilon_0 (E_{2n} - E_{1n}) = \sigma_s \Rightarrow \mathbf{n} \cdot \left[ \varepsilon_0 (\mathbf{E}_{2} - \mathbf{E}_{1}) \right] = \sigma_s \]

2. Continuity of Tangential \( \mathbf{E} \)

\[ \oint (\mathbf{E} \cdot d\mathbf{s}) = (E_{1t} - E_{2t}) dl = 0 \Rightarrow E_{1t} - E_{2t} = 0 \]

\[ \mathbf{n} \times (\mathbf{E}_{2} - \mathbf{E}_{1}) = 0 \]

Equivalent to \( \Phi_1 = \Phi_2 \) along boundary
3. Normal $\mathbf{H}$

$$\int_S \mu_0 \mathbf{H} \cdot \mathbf{d}a = 0$$

\[ \mu_0 \left( H_{an} - H_{bn} \right) A = 0 \]

\[ H_{an} = H_{bn} \]

\[ \mathbf{n} \cdot \left[ \mathbf{H}_a - \mathbf{H}_b \right] = 0 \]

4. Tangential $\mathbf{H}$

$$\oint_C \mathbf{H} \cdot \mathbf{ds} = \int_S \mathbf{j} \cdot \mathbf{da} + \frac{d}{dt} \int_S \varepsilon_0 \mathbf{E} \cdot \mathbf{da}$$

\[ H_{bt} \Delta \chi - H_{at} \Delta \chi = K \Delta \chi \]

\[ H_{bt} - H_{at} = K \]

\[ \mathbf{n} \times \left[ \mathbf{H}_a - \mathbf{H}_b \right] = \mathbf{K} \]

5. Conservation of Charge Boundary Condition

$$\oint_S \mathbf{j} \cdot \mathbf{da} + \frac{d}{dt} \int_V \rho dV = 0$$

\[ \mathbf{n} \cdot \left[ \mathbf{j}_a - \mathbf{j}_b \right] + \frac{\partial}{\partial t} \sigma_s = 0 \]
6. Electric Field from a Sheet of Surface Charge

a. Electric Field from a Line Charge

An infinitely long uniform distribution of line charge only has a radially directed electric field because the z components of the electric field are canceled out by symmetrically located incremental charge elements as shown above.
\[ dE_r = \frac{dq}{4\pi\varepsilon_0 \left( r^2 + z^2 \right)} \cos \theta = \frac{\lambda_0 rdz}{4\pi\varepsilon_0 \left( r^2 + z^2 \right)^{\frac{3}{2}}} \]

\[ E_r = \int_{z=-\infty}^{+\infty} dE_r = \frac{\lambda_0 r}{4\pi\varepsilon_0} \int_{z=-\infty}^{+\infty} \frac{dz}{\left( r^2 + z^2 \right)^{\frac{3}{2}}} \]

\[ = \frac{\lambda_0 r}{4\pi\varepsilon_0} \left. \frac{z}{r^2 \left( z^2 + r^2 \right)^{\frac{1}{2}}} \right|_{z=-\infty}^{+\infty} \]

\[ = \frac{\lambda_0}{2\pi\varepsilon_0 r} \]

Another way: Gauss’ Law

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\[ \int_S \varepsilon_0 \vec{E} \cdot d\vec{a} = \varepsilon_0 E_r 2\pi rL = \lambda_0 L \]

\[ E_r = \frac{\lambda_0}{2\pi\varepsilon_0 r} \]
b. Electric Field from a Sheet Charge

(a) The electric field from a uniformly surface charged sheet of infinite extent is found by summing the contributions from each incremental line charge element. Symmetrically placed line charge elements have x field components that cancel, but y field components that add. (b) Two parallel but oppositely sheets of surface charge have fields that add in the region between the sheets but cancel outside. (c) The electric field from a volume charge distribution is obtained by summing the contributions from each incremental surfacex charge element:

\[
\begin{align*}
\text{d}E_y &= \frac{\text{d}\lambda}{2\pi \varepsilon_0 \left(x^2 + y^2 \right)^{1/2}} \cos \theta = \frac{\sigma_0 y \text{d}x}{2\pi \varepsilon_0 \left(x^2 + y^2 \right)} \\
E_y &= \int_{x=-\infty}^{+\infty} \text{d}E_y = \frac{\sigma_0 Y}{2\pi \varepsilon_0} \int_{x=-\infty}^{+\infty} \frac{\text{d}x}{x^2 + y^2} \\
&= \frac{\sigma_0 Y}{2\pi \varepsilon_0} \left[ \tan^{-1} \frac{x}{y} \right]_{-\infty}^{+\infty}
\end{align*}
\]
\[
\begin{align*}
\sigma_0 & \quad y > 0 \\
-\frac{\sigma_0}{2\varepsilon_0} & \quad y < 0
\end{align*}
\]

Checking Boundary condition at \(y=0\)

\[
E_y (y = 0^+) - E_y (y = 0^-) = \frac{\sigma_0}{\varepsilon_0}
\]

\[
\frac{\sigma_0}{2\varepsilon_0} - \left(-\frac{\sigma_0}{2\varepsilon_0}\right) = \frac{\sigma_0}{\varepsilon_0}
\]

c. Two sheets of Surface Charge (Capacitor)

\[
\bar{E}_1 = \begin{cases} 
\frac{\sigma_0}{2\varepsilon_0} \bar{I}_y & y > -a \\
-\frac{\sigma_0}{2\varepsilon_0} \bar{I}_y & y < -a
\end{cases}, \quad 
\bar{E}_2 = \begin{cases} 
-\frac{\sigma_0}{2\varepsilon_0} \bar{I}_y & y > a \\
\frac{\sigma_0}{2\varepsilon_0} \bar{I}_y & y < a
\end{cases}
\]

\[
\bar{E} = \bar{E}_1 + \bar{E}_2 = \begin{cases} 
\frac{\sigma_0}{\varepsilon_0} \bar{I}_y & |y| < a \\
0 & |y| > a
\end{cases}
\]
7. Magnetic Field from a Sheet of Surface Current

A uniform surface current of infinite extent generates a uniform magnetic field oppositely directed on each side of the sheet. The magnetic field is perpendicular to the surface current but parallel to the plane of the sheet. (b) The magnetic field due to a slab of volume current is found by superimposing the fields due to incremental surface currents. (c) Two parallel but oppositely directed surface current sheets have fields that add in the region between the sheets but cancel outside the sheet. (d) The force on a current sheet is due to the average field on each side of the sheet as found by modeling the sheet as a uniform volume current distributed over an infinitesimal thickness $\Delta$.

From a line current $I$

$$H_\phi = \frac{I}{2\pi r}$$

$$\vec{I}_x = -\sin \phi \vec{i}_x + \cos \phi \vec{i}_y$$

Thus from 2 symmetrically located line currents

$$dH_x = \frac{dI}{2\pi \left(x^2 + y^2 \right)^{1/2} \left(-\sin \phi \right)}$$
\[ H_x = -\frac{K_0}{2\pi} y \int_{x=-\infty}^{+\infty} \frac{dx}{x^2 + y^2} \]

\[ = -\frac{K_0 y}{2\pi} \frac{1}{y} \tan^{-1} x \bigg|_{x=-\infty}^{+\infty} \]

\[ = \begin{cases} 
-\frac{K_0}{2} & y > 0 \\
+\frac{K_0}{2} & y < 0 
\end{cases} \]

Check boundary condition at \( y=0 \):

\[ H_x (y = 0_+) - H_x (y = 0_-) = -K_0 \]

\[ -\frac{K_0}{2} - \left( \frac{K_0}{2} \right) = -K_0 \]