Spectral Measurements

Case A: Bandwidth exceeds that of available amplifiers

1) Extreme bandwidth: use multiple receivers and antennas

2) If signal large compared to detector noise, detect directly or split frequencies and then detect

3) Use passive frequency splitters before amplification or detection
Spectral Measurements

Case B: Bandwidth permits amplification
1) Amplify before either detection or further frequency splitting

Case C: Bandwidth permits digital spectral analysis
1) If computer resources permit, compute

$$\left\langle \left| \frac{V(f)}{N} \right|^2 \right\rangle_M \left( \sim N \log_2 N \text{ multiplys per } N \text{-point transform} : \text{average } M \text{ spectra} \right)$$
Resolution $\Delta f \geq 2B/N$

2) Or 1-bit (or n-bit) $\phi_N(\tau) \leftrightarrow \Phi_N(f)(N \text{ samples})$

(Permits $\sim \times 100$ more B per cm$^2$ silicon)
Examples of Passive Multichannel Filters

1. Circuits

2. Waveguides
Examples of Passive Multichannel Filters

3. Prism

![Prism Diagram]

4. Diffraction grating

![Diffraction Grating Diagram]

5. Cascaded Dichroics

![Cascaded Dichroics Diagram]
Digital spectral analysis example: autocorrelation

Possible analog implementation:

\[ \hat{\phi}_V(\tau) \text{ is based on:} \]
1) max lag = \( \tau_{\text{max}} = NT \)
2) sample lag, T sec
3) finite integration time \( \tau \gg \tau_{\text{max}} \)
Resolution of autocorrelation analysis

1) \( \hat{\Phi}_v(\tau) = \phi_y(\tau) \ast W(\tau) \)

\( \Leftrightarrow |\tau| < \tau_M \)

\[ \therefore \hat{\Phi}_v(f) = \Phi_v(f) \ast W(f) \]

Thus

\[ \Phi_v(f) \]
2) $\hat{\phi}_V(\tau) = \phi_v(\tau) \cdot i(t)$

$\hat{\Phi}_V(f) = \Phi_v(f) \ast l(f)$

"Aliasing" is spectral overlap

3) Finite averaging time $\tau$ adds noise to $\hat{\phi}_V(\tau), \hat{\Phi}_V(f)$
Autocorrelation of hard-clipped signals

\[ \hat{v}_v(\tau) = \int \hat{v}(t) \hat{v}(t+\tau) \, dt \]

where \( \hat{v}(t) \) is the hard-clipped signal and \( \tau \) is the delay. The diagram shows a block diagram of the autocorrelation process, including the hard clipping, A/D conversion, and delay line.
Analysis of 1-bit autocorrelation

Let \( x(t_1) \overset{\Delta}{=} x_1, \ x(t_2) \overset{\Delta}{=} x_2, \ \text{sgn} \ x \overset{\Delta}{=} \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases} \) where \( x_1, x_2 \) are JGRVZM

\[
\phi_x(\tau) = E[\text{sgn} \ x_1 \ \text{sgn} \ x_2] = \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sgn} \ x_1 \ \text{sgn} \ x_2 \left[ \frac{1}{2\pi(1-\rho)^{1/2}} e^{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}} \right] dx_1 dx_2
\]

where \( \rho(\tau) \overset{\Delta}{=} x_1 x_2 \equiv \phi_v(\tau), \ \tau = t_2 - t_1 \)

\[
\phi_x(\tau) = 2 \int_{0}^{\infty} \int_{0}^{\infty} p(x_1, x_2) dx_1 dx_2 - 2 \int_{-\infty}^{0} \int_{0}^{\infty} p(x_1, x_2) dx_1 dx_2 \\
= 4 \int_{0}^{\infty} \int_{0}^{\infty} p(x_1, x_2) dx_1 dx_2 - 1 \left\{ \text{Note: } 2 \int_{0}^{\infty} + 2 \int_{-\infty}^{0} = 1 \right\}
\]
Power spectrum for 1-bit signal

Change variables

\[ \begin{align*}
  x_1 &= r \cos \theta \\
  x_2 &= r \sin \theta \\
  dx_1 dx_2 &= rdr \, d\theta
\end{align*} \]

\[
\phi_x(\tau) = 4 \int_0^{\pi/2} \int_0^\infty \frac{r^2}{2} \frac{1}{2\pi(1 - \rho^2)^{1/2}} e^{-(r^2/2) \left( \frac{1 - \rho \sin 2\theta}{1 - \rho^2} \right)} - 1
\]

\[
= 4 \int_0^{\pi/2} d\theta \frac{(1 - \rho^2)^{1/2}}{2\pi(1 - \rho \sin 2\theta)} - 1
\]
Power spectrum for 1-bit signal

\[
\phi_x(\tau) = 4 \frac{(1-\rho)^{1/2}}{4\pi} \int_0^\pi \frac{1}{1-\rho \sin \phi} \, d\phi - 1 = 4 \left\{ \frac{1}{2\pi} \left( \frac{\pi}{2} + \sin^{-1} \rho \right) \right\} - 1
\]

Let \( \phi \equiv 2\theta \)

\[
\hat{\phi}_v(\tau) \equiv \hat{\rho} = \sin \left( \frac{\pi}{2} \hat{\phi}_x(\tau) \right)
\]

Where \( \hat{\phi}_x(\tau) = \langle (\text{sgn } v(t))(\text{sgn } v(t - \tau)) \rangle_T \)

Note: \( \hat{\rho} \) has bias if \( b \) not exact

Spectral response & sensitivity: autocorrelation receiver

\[ \sigma(f)_{\text{rms}} \approx \frac{\alpha \beta T_{\text{eff}}}{\sqrt{\tau \Delta f}} \sqrt{1 - \frac{\Delta f}{B}} \; ; \; \beta \approx 1.6 \]

(S. Weinreb empirical result, MIT EE PhD thesis, 1963)

“Apodizing” weighting functions:

<table>
<thead>
<tr>
<th>Weighting Function</th>
<th>( \alpha )</th>
<th>( \frac{\Delta f}{N} )</th>
<th>Sidelobe</th>
</tr>
</thead>
<tbody>
<tr>
<td>uniform</td>
<td>1.099</td>
<td>0.60 ( f_s )</td>
<td>-7 dB</td>
</tr>
<tr>
<td>raised cosine</td>
<td>0.87</td>
<td>( f_s / N )</td>
<td>-16 dB</td>
</tr>
<tr>
<td>blackman</td>
<td>0.69</td>
<td>1.13 ( f_s / N )</td>
<td>-29 dB</td>
</tr>
</tbody>
</table>

Note trade between spectral resolution, sidelobes in \( \Phi(f) \) and \( \Delta T_{\text{rms}} \)
If $N$ delay-line taps, how many spectral samples $N_s$?

Say uniform weighting of $\phi(\tau)$:

Then $B = N_s \cdot \Delta f = N_s \cdot (1/2\tau_M)$ where spectral resolution $\Delta f \approx 1/2\tau_m$ for orthogonal channels from boxcar $W(\tau)$

$W(f)$ for adjacent channel

$N_s = 2\tau_M B = 2 NT B (T = 1/2B$ at nyquist rate$) = N($# taps$)$

In practice: raised cosine widens $\Delta f$ by $1/0.6 \approx 1.7$, so $N_s \approx N/1.7$
Types of “power”
Delivered
Available
Exchangeable

\[ v(t) \triangleq R_e \left\{ V e^{j\omega t} \right\} = R_e \{ V \} \cos \omega t + I_m \{ V \} \sin \omega t \]

\[ P_{\text{delivered}} \triangleq \frac{1}{2} R_e \left\{ V^* \right\} (\Delta P_D) \]

\[ P_{\text{available}} \triangleq \max P_D, \text{ i.e., if } Z_L = Z_g^* \]
Delivered and Available Power

\[ P_{\text{delivered}} = \frac{1}{2} R_e \left\{ V I^* \right\} (\Delta P_D) \]

\[ P_{\text{available}} = \max P_D, \text{ i.e., if } Z_L = Z_g^* \]

If: \( R_e \{ Z_g \} > 0 \)

If: \( R_e \{ Z_g \} < 0 \)

\[ P_{\text{exchangeable}} = \left| P_D \right|_{Z_L = Z_g^*} \rightarrow \text{finite - power option} \]
Definition of Gain

\[ G_\text{power} \quad (= \quad G_p) \quad \triangleq \quad \frac{P_{D_2}}{P_{D_1}} \]

\[ G_\text{available} \quad (= \quad G_A) \quad \triangleq \quad \frac{P_{A_2}}{P_{A_1}} \]

\[ G_\text{transducer power} \quad (G_T) \quad \triangleq \quad \frac{P_{D_2}}{P_{A_1}} \]

\[ G_\text{insertion} \quad (= \quad G_I) \quad \triangleq \quad \frac{P_{D_2}}{P_{D_1}} \quad \text{with amplifier} \]

\[ G_\text{exchangeable} \quad (=G_E) \quad \triangleq \quad \frac{P_{E_2}}{P_{E_1}} \quad \text{without amplifier} \]

Note: \( G_A, G_E \) don’t depend on \( Z_L \)
depend on \( Z_g \) (via \( P_{E_2} \))
Definition: Signal-to-Noise Ratio (SNR)

First define:

\[ WH_z^{-1} \]

\[ N_1 = \text{exchangeable noise power spectrum @ Port 1} \]
\[ N_2 = \text{same, at 2} \]
\[ S_1 = \text{exchangeable signal power spectrum @ Port 1} \]
\[ S_2 = \text{same, at 2} \]

Define \( SNR_1 \triangleq S_1/N_1 ; \) \( SNR_2 \triangleq S_2/N_2 \)
Definition: Noise Figure F

\[ F \equiv \frac{S_{1}/N_1}{S_{2}/N_2} \]

\[ \text{where } N_1 \equiv kT_o, \quad T_o \equiv 290 \text{ K} \]


\[ S_2 = G_E S_1 \text{ (see definition of } G_E) \]

\[ N_2 = G_E N_1 + N_{2T} \text{ “transducer noise”} \]

\[ \therefore F = \frac{S_1/N_1}{G S_1/(G N_1 + N_{2T})} = 1 + \frac{N_{2T}}{N_1 G} \]

(\text{let } G \equiv G_E)

\[ \therefore F - 1 = \frac{N_{2T}}{N_1 G} \equiv \frac{kT_R G}{kT_0 G} = \frac{T_R}{T_0} \]

“receiver noise temperature”

“excess noise figure”
“Excess noise” corresponds to “receiver noise temperature $T_R$”

Examples:

- $T_R = 0^\circ K \implies F = 1 + \frac{T_R}{T_o} = 1$ (F = 0 dB)
- $T_R = 290^\circ K \implies F = 2$ (F = 3 dB)
- $T_R = 1500^\circ K \implies F \approx 6$ (F $\approx 7.5$ dB)