Gaussian Wavepacket supplement to 6.728 notes
S. D. Senturia

We start with a stationary (unnormalized) Gaussian wavepacket
\[ e^{-x^2/2L^2} \]  
for which the Fourier transform is
\[ \sqrt{2\pi L^2} e^{-q^2L^2/2} \]  

Performing the superposition of deBroglie plane waves, we find
\[ \Psi(x,t) = \sqrt{2\pi L^2} \int_{-\infty}^{\infty} e^{-q^2L^2/2} e^{i(qx - \hbar q^2t/2m)} \frac{dq}{2\pi} \]  

This can be written
\[ \Psi(x,t) = \sqrt{2\pi L^2} \int_{-\infty}^{\infty} e^{-\left[q^2(L^2/2 + i\hbar t/2m) - iqx\right]} \frac{dq}{2\pi} \]  

Recalling from high school algebra the trick of “completing the square”, the exponent can be written
\[ -\left(q\sqrt{L^2/2 + i\hbar t/2m} - \frac{ix}{2\sqrt{L^2/2 + i\hbar t/2m}}\right)^2 - \frac{x^2}{4\left(L^2/2 + i\hbar t/2m\right)} \]  

The last term in the exponent doesn’t depend on q, so it comes outside the integral, leading to
\[ \Psi(x,t) = \sqrt{2\pi L^2} e^{-\left[x^2/2L^2\left(1 + i\hbar t/mL^2\right)\right]} \int_{-\infty}^{\infty} e^{-\left[q\sqrt{L^2/2 + i\hbar t/2m} - \frac{ix}{2\sqrt{L^2/2 + i\hbar t/2m}}\right]^2} \frac{dq}{2\pi} \]

The integral is now a standard one, having the value
\[ \frac{\sqrt{\pi}}{2\pi \sqrt{L^2/2 + i\hbar t/2m}} \]  

This leads to the final result in Eq. 4.18 of the notes, for the case \( k = 0 \).