6.730 Physics for Solid State Applications

Lecture 4: Vibrations in Solids

Outline

- Review Lecture 3
- Sommerfeld Theory of Metals
- 1-D Elastic Continuum
- 1-D Lattice Waves
- 3-D Elastic Continuum
- 3-D Lattice Waves
Balance equation for forces on electrons:

\[
m \frac{d\mathbf{v}(r, t)}{dt} = -m \frac{\mathbf{v}(r, t)}{\tau} - e \left[ \mathbf{E}(r, t) + \mathbf{v}(r, t) \times \mathbf{B}(r, t) \right]
\]

DRAG FORCE

LORENTZ FORCE

In steady-state when \( B=0 \):

\[
\mathbf{v} = -\frac{e\tau}{m} \mathbf{E}_{DC}
\]

\[
\mathbf{J} = -ne\mathbf{v} = \frac{ne^2\tau}{m} \mathbf{E}_{DC}
\]

\[
\mathbf{J} = \sigma \mathbf{E}_{DC} \quad \text{and} \quad \sigma = \frac{ne^2\tau}{m}
\]
Density of States

\[ n = \frac{N}{V} = \int_{-\infty}^{\infty} \frac{1}{1 + e^{(E_\mathbf{k} - \mu)/k_BT}} 2 \frac{d^3 k}{(2\pi)^3} \]

\[ n = \int_{-\infty}^{\infty} g(E) f(E-\mu) dE = \int_{-\infty}^{\infty} g(E) \frac{1}{1 + e^{(E - \mu)/k_BT}} dE \]
Microscopic Variables for Electrical Transport

Balance equation for energy of electrons:

\[
\frac{dE}{dt} = -\frac{\Delta E}{\tau} + IV
\]

In steady-state:

\[
\Delta E = \tau IV
\]

In the continuum models, we assume that electron scattering is sufficiently fast that all the energy pumped into the electrons is randomized; all additional energy heats the electrons.

How do we relate \( \Delta E \) and \( T \)?
Specific Heat and Heat Capacity

Again assume that the heat and change in internal energy are the same:

\[ c_V = \left( \frac{dQ}{dT} \right)_V = \left( \frac{dE_{\text{total}}}{dT} \right)_V \]  

(heat capacity)

Take constant volume since this ensures none of the extra energy is going into work (think ideal gas)

\[ C_V = \frac{1}{V} \frac{d}{dT} \left( \frac{3}{2} N k_B T \right) = \frac{3}{2} n k_B \]  

(specific heat)

\[ C_v = 2 \times 10^6 \frac{\text{erg}}{\text{cm}^3 \cdot \text{K}} = 11 \frac{\text{Joule}}{\text{mole} \cdot \text{K}} \]

Specific heat is independent of temperature…Law of Dulong and Petit
Specific heat is independent of temperature...NOT TRUE!

To get this correct we will need to (a) quantize electron energy levels, (b) introduce discreteness of lattice and (c) the heat capacity of lattice.
Outline

• Review Lecture 3
• **Sommerfeld Theory of Metals**
• 1-D Elastic Continuum
• 1-D Lattice Waves
• 3-D Elastic Continuum
• 3-D Lattice Waves
Low Temperature Specific Heat of the Free Electron Gas
Sommerfeld Approximation

\[
\frac{\Delta E}{V} \approx \frac{[g(E_{F0}) k_B T]}{\text{excited states}} \frac{k_B T}{\text{increase in energy}}
\]

\[
C_V = \frac{\partial (\Delta E/V)}{\partial T} \approx 2g(E_{F0}) k_B^2 T
\]

\[
C_V \approx \left( \frac{3}{2} n k_B \right) \left( \frac{2 k_B T}{E_{F0}} \right) \approx \left( \frac{3}{2} n k_B \right) (100 - 5000)
\]
Conductivity of the Free Electron Gas

Sommerfeld Approximation

\[ v_d = (-e\tau/m)E_{DC} \]

\[ v = v_F - \frac{e\tau}{m}E_{DC} \]

\[ E = \frac{1}{2}mv^2 \approx \frac{1}{2}mv_F^2 + e\tau v \cdot E_{DC} \]

\[ \Delta E = e\tau v_F |E_{DC}| \]

\[ J = -e(\delta n)v_F \]

\[ \delta n \approx g(E_F)\Delta E \]

Only electrons near \( E_F \) contribute to current!
Conductivity of the Free Electron Gas
Sommerfeld Approximation

\[ \Delta E = e \tau v_F |E_{DC}| \]
\[ J = -e (\delta n) v_F \]
\[ \delta n \approx g(E_F) \Delta E \]

\[ J = e^2 \tau v_F^2 g(E_F) |E_{DC}| \]
\[ \sigma = e^2 v_F^2 \tau g(E_F) \]
\[ \sigma \approx \frac{n e^2 \tau}{m} \]

Sommerfeld recovers the phenomenological results!
Sommerfeld Expansion

\[ f(E-\mu) = \lim_{T \to 0} \frac{1}{1 + e^{(E-\mu)/k_B T}} = 1 - u(E-\mu) \]

\[ f'(E - \mu) = -\delta(E - E_{F0}) \]

\[
\int_{-\infty}^{\infty} f(E-\mu)H(E)dE = \int_{-\infty}^{\mu} H(E)dE + \frac{\pi^2}{6} (k_B T)^2 H'(\mu) + O \left( \frac{k_B T}{E_{F0}} \right)^4
\]

\[
\int_{-\infty}^{\mu} H(E)dE = \int_{-\infty}^{E_{F0}} H(E)dE + (\mu - E_{F0})H(E_{F0}) + O \left( \frac{k_B T}{E_{F0}} \right)^4
\]
Sommerfeld Expansion for Electron Density

\[ n = \int_{0}^{E_{F0}} g(E) dE + \left[ (\mu - E_{F0}) g(E_{F0}) + \frac{\pi^2}{6} (k_B T)^2 g'(E_{F0}) \right] \]

\[ \mu = E_{F0} \left\{ 1 - \frac{\pi^2}{6} \left( \frac{(k_B T)^2}{E_{F0}} \right) \frac{g'(E_{F0})}{g(E_{F0})} \right\} \]

\[ \mu = E_{F0} \left\{ 1 - \frac{1}{3} \left( \frac{\pi k_B T}{2E_{F0}} \right)^2 \right\} \]
Sommerfeld Expansion for Electron Energy

\[
\frac{E}{V} = \int_{-\infty}^{\infty} E g(E) f(E - \mu) dE \\
= \int_0^{E_{F0}} E g(E) dE + E_{F0} \left[ (\mu - E_{F0}) g(E_{F0}) + \frac{\pi^2}{6} (k_B T)^2 g'(E_{F0}) \right] \\
+ \frac{\pi^2}{6} (k_B T)^2 g(E_{F0}) + O(T^4)
\]

\[
\frac{E}{V} = \int_0^{E_{F0}} E g(E) dE + \frac{\pi^2}{6} (k_B T)^2 g(E_{F0}) \\
= \frac{3}{5} E_F n + \frac{\pi^2}{6} (k_B T)^2 g(E_{F0})
\]

\[
C_V = \left. \frac{\partial ((E/V))}{\partial T} \right|_{V,N} = \frac{\pi^2}{3} k_B^2 T g(E_{F0}) = \gamma T
\]
Specific Heat Measurements

To get this correct we will need to (a) quantize electron energy levels, (b) introduce discreteness of lattice and (c) the heat capacity of lattice.
Density of States is the Central Character in this Story

Goal: Calculate electrical properties (e.g. resistance) for solids

Approach:
In the end calculating resistance boils down to calculating the electronic energy levels and wavefunctions; to knowing the bandstructure

You will be able to relate a bandstructure to macroscopic parameters for the solid

\[ \sigma = e^2 v_F^2 \tau g(E_F) \]

\[ C_V = \frac{\partial ((E/V))}{\partial T} \bigg|_{V,N} = \frac{\pi^2}{3} k_B T g(E_{F0}) = \gamma T \]
Outline

- Review Lecture 3
- Sommerfeld Theory of Metals
- **1-D Elastic Continuum**
  - 1-D Lattice Waves
- 3-D Elastic Continuum
- 3-D Lattice Waves
1-D Elastic Continuum
Stress and Strain

uniaxial loading

\[ T_{xx} = \frac{F_x}{A} \left[ \frac{N}{m^2} \right] \]

Stress:

Strain:

\[ \delta(dx) = u_x(x + dx) - u_x(x) \]

Normal strain:

\[ E_{xx} = \frac{\delta(dx)}{dx} = \frac{\partial u_x}{\partial x} \]

If \( u_x \) is uniform there is no strain, just rigid body motion.
1-D Elastic Continuum
Young’s Modulus

\[ T_{xx} = E_Y E_{xx} \]

Young’s Modulus For Various Materials (GPa)
from Christina Ortiz

<table>
<thead>
<tr>
<th>METALS:</th>
<th></th>
</tr>
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<tbody>
<tr>
<td>Tungsten (W)</td>
<td>406</td>
</tr>
<tr>
<td>Chromium (Cr)</td>
<td>289</td>
</tr>
<tr>
<td>Beryllium (Be)</td>
<td>200 - 289</td>
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<tr>
<td>Nickel (Ni)</td>
<td>214</td>
</tr>
<tr>
<td>Iron (Fe)</td>
<td>196</td>
</tr>
<tr>
<td>Low Alloy Steels</td>
<td>200 - 207</td>
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<tr>
<td>Stainless Steels</td>
<td>190 - 200</td>
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<tr>
<td>Cast Irons</td>
<td>170 - 190</td>
</tr>
<tr>
<td>Copper (Cu)</td>
<td>124</td>
</tr>
<tr>
<td>Titanium (Ti)</td>
<td>116</td>
</tr>
<tr>
<td>Brasses and Bronzes</td>
<td>103 - 124</td>
</tr>
<tr>
<td>Aluminum (Al)</td>
<td>69</td>
</tr>
</tbody>
</table>

| PINE WOOD (along grain): | 10 |

<table>
<thead>
<tr>
<th>POLYMERS:</th>
<th></th>
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<tbody>
<tr>
<td>Polymides</td>
<td>3 - 5</td>
</tr>
<tr>
<td>Polyesters</td>
<td>1 - 5</td>
</tr>
<tr>
<td>Nylon</td>
<td>2 - 4</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>3 - 3.4</td>
</tr>
<tr>
<td>Polyethylene</td>
<td>0.2 - 0.7</td>
</tr>
<tr>
<td>Rubbers / Biological Tissues</td>
<td>0.01 - 0.1</td>
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<table>
<thead>
<tr>
<th>CERAMICS GLASSES AND SEMICONDUCTORS</th>
<th></th>
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<tbody>
<tr>
<td>Diamond (C)</td>
<td>1000</td>
</tr>
<tr>
<td>Tungsten Carbide (WC)</td>
<td>450 - 650</td>
</tr>
<tr>
<td>Silicon Carbide (SiC)</td>
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</tr>
<tr>
<td>Aluminum Oxide (Al₂O₃)</td>
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<td>Beryllium Oxide (BeO)</td>
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<td>Magnesium Oxide (MgO)</td>
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<td>Zirconium Oxide (ZrO)</td>
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<td>Mullite (Al₆Si₂O₁₃)</td>
<td>145</td>
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<tr>
<td>Silicon (Si)</td>
<td>107</td>
</tr>
<tr>
<td>Silica glass (SiO₂)</td>
<td>94</td>
</tr>
<tr>
<td>Soda-lime glass (Na₂O - SiO₂)</td>
<td>69</td>
</tr>
</tbody>
</table>
Dynamics of 1-D Continuum

1-D Wave Equation

Net force on incremental volume element:

\[ f_x = \left[ T_{xx}(x + dx) - T_{xx}(x) \right] dy \, dz \]

\[ m \frac{\partial^2 u_x}{\partial t^2} = \left[ T_{xx}(x + dx) - T_{xx}(x) \right] dy \, dz \]

\[ \rho \frac{\partial^2 u_x}{\partial t^2} \, dx \, dy \, dz = \left[ T_{xx}(x + dx) - T_{xx}(x) \right] dy \, dz \]

\[ \rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial T_{xx}}{\partial x} \]
Dynamics of 1-D Continuum

1-D Wave Equation

\[ \rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial T_{xx}}{\partial x} \]

\[ T_{xx} = E_Y E_{xx} \]

\[ E_{xx} = \frac{\partial u_x}{\partial x} \]

\[ \rho \frac{\partial^2 u_x}{\partial t^2} = E_Y \frac{\partial^2 u_x}{\partial x^2} \]

\[ \frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u_x}{\partial t^2} \]

\[ c = \sqrt{\frac{E_Y}{\rho}} \]

Velocity of sound, \( c \), is proportional to stiffness and inverse prop. to inertia
Dynamics of 1-D Continuum
1-D Wave Equation Solutions

\[ \frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u_x}{\partial t^2} \]

Clamped Bar: Standing Waves

\[ u_x(x, t) = A_{\pm} \sin(kx) \exp(i\omega t) \quad \omega = ck \]

\[ u_{x,m,\pm}(x, t) = A_{m,\pm} \sin \left( \frac{m\pi x}{L} \right) \exp \left( \pm i \frac{m\pi c}{L} t \right) \]

\[ k = \frac{m\pi}{L} \quad \text{for} \quad m = 1, 2, \ldots \]
Dynamics of 1-D Continuum
1-D Wave Equation Solutions

\[ \frac{\partial^2 u_x}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u_x}{\partial t^2} \]

Periodic Boundary Conditions: Traveling Waves

\[ u_x(x, t) = A_{\pm} \exp(ikx) \exp(i\omega t) \quad \omega = ck \]

\[ u_{x,n,\pm}(x, t) = B_{n,\pm} \exp \left( \pm i \frac{2n\pi x}{L} (x \pm ct) \right) \]

\[ k = \frac{2n\pi}{L} \quad \text{for} \quad n = \pm 1, \pm 2, \ldots \]
3-D Elastic Continuum
Volume Dilatation

dx \rightarrow dx + \delta(dx) = dx + E_{xx}dx = dx(1 + E_{xx})

dy \rightarrow dy(1 + E_{yy})

dz \rightarrow dz(1 + E_{zz})

\[
e = \frac{\delta V}{V} = \frac{dx(1 + E_{xx})dy(1 + E_{yy})dz(1 + E_{zz}) - dxdydz}{dxdydz}
\]

\[e = E_{xx} + E_{yy} + E_{zz}\]

Volume change is sum of all three normal strains
3-D Elastic Continuum
Poisson’s Ratio

\[ E_{xx} = \frac{\partial u_x}{\partial x} \quad E_{yy} = \frac{\partial u_y}{\partial y} \quad E_{zz} = \frac{\partial u_z}{\partial z} \]

\[ e = E_{xx} + E_{yy} + E_{zz} = \nabla \cdot \mathbf{u}(\mathbf{r}) \]

\( \nu \) is Poisson’s Ratio – ratio of lateral strain to axial strain

\[ E_{yy} = E_{zz} = -\nu E_{xx} \]

\[ e = E_{xx}(1 - 2\nu) \]

Poisson’s ratio can not exceed 0.5, typically 0.3
3-D Elastic Continuum
Poisson’s Ratio Example

Aluminum: $E_Y = 68.9$ GPa, $\nu = 0.35$
3-D Elastic Continuum
Poisson’s Ratio Example

Aluminum: $E_Y = 68.9$ GPa, $\nu = 0.35$

$$T_{xx} = \frac{F_x}{A} = \frac{5 \times 10^3}{\pi (10 \times 10^{-3})^2} = -15.9 \text{MPa}$$

$$E_{xx} = \frac{T_{xx}}{E_Y} = \frac{-15.9 \times 10^6}{68.9 \times 10^9} = -0.231 \times 10^{-3}$$

$$E_{xx} = \frac{\Delta l}{l} = -0.231 \times 10^{-3}$$

$$\Delta l = -0.0173 \text{mm}$$
3-D Elastic Continuum
Poisson’s Ratio Example

Aluminum: \( E_Y = 68.9 \text{ GPa}, \ \nu = 0.35 \)

\[
\begin{align*}
T_{xx} &= \frac{F_x}{A} = \frac{5 \times 10^3}{\pi (10 \times 10^{-3})^2} = -15.9 \text{MPa} \\
E_{xx} &= \frac{T_{xx}}{E_Y} = \frac{-15.9 \times 10^6}{68.9 \times 10^9} = -0.231 \times 10^{-3} \\
E_{xx} &= \frac{\Delta l}{l} = -0.231 \times 10^{-3} \\
\Delta l &= -0.0173 \text{mm}
\end{align*}
\]

\[
E_{trns} = -\nu E_{xx} = -0.35 E_{xx} = 0.081 \times 10^{-3}
\]

\[
E_{trns} = \frac{\Delta d}{d} \quad \Delta d = +0.001617 \text{mm}
\]
3-D Elastic Continuum
Shear Strain

Shear loading

Shear plus rotation

Pure shear

Pure shear strain

\[ \phi = E_{xy} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \]

Shear stress

\[ T_{xy} = G \, 2\phi = 2G E_{xy} \quad \text{G is shear modulus} \]
3-D Elastic Continuum
Stress and Strain Tensors

\[ E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

\[ E_{xx} = \frac{\partial u_x}{\partial x} \]

\[ E_{xy} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \]

\[ e = \sum_{k=1}^{3} E_{kk} \]

For most general isotropic medium,

\[ T = \lambda e I + 2\mu E \]

Initially we had three elastic constants: \( E_Y, G, e \)

Now reduced to only two: \( \lambda, \mu \)
3-D Elastic Continuum
Stress and Strain Tensors

\[ T_{ij} = \lambda \epsilon \delta_{ij} + 2\mu E_{ij} \]

If we look at just the diagonal elements

\[ \sum_{k=1}^{3} T_{kk} = 3\lambda \epsilon + 2\mu \epsilon \]

\[ e = \frac{1}{3\lambda + 2\mu} \sum_{k=1}^{3} T_{kk} \]

Inversion of stress/strain relation:

\[ E_{ij} = \frac{1}{2\mu} \left[ T_{ij} - \frac{\lambda}{3\lambda + 2\mu} \left( \sum_{k} T_{kk} \right) \delta_{ij} \right] \]
3-D Elastic Continuum
Example of Uniaxial Stress

\[ E_{ij} = \frac{1}{2\mu} \left[ T_{ij} - \frac{\lambda}{3\lambda + 2\mu} \left( \sum_k T_{kk} \right) \delta_{ij} \right] \]

\[ E_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} T_{11} \]

\[ E_{22} = E_{33} = -\frac{\lambda}{2(\lambda + \mu)} E_{11} \]
Dynamics of 3-D Continuum

3-D Wave Equation

Net force on incremental volume element:

\[ F = \int_V f \, dx \, dy \, dz \]

\[ F = \int_V \rho \frac{\partial^2 u}{\partial t^2} \, dx \, dy \, dz \]

\[ f = \rho \frac{\partial^2 u}{\partial t^2} \]

Total force is the sum of the forces on all the surfaces
Dynamics of 3-D Continuum

3-D Wave Equation

Net force in the x-direction:

\[ F_x = \sum_{\text{surfaces}} (T_{xx} \, dA_x + T_{xy} \, dA_y + T_{xz} \, dA_z) \]

\[ \sum_{\text{surface}} T_{xx} \, dA_x = \frac{T_{xx}(x + dx) - T_{xx}(x)}{dx} \, dx \, dy \, dz \]

\[ \sum_{\text{surface}} T_{xx} \, dA_x = \frac{\partial T_{xx}}{\partial x} \, dx \, dy \, dz \]

\[ F_x = \int \int \int \left[ \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right] \, dx \, dy \, dz \]
Dynamics of 3-D Continuum

3-D Wave Equation

\[ F_x = \iiint \left[ \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right] \, dx \, dy \, dz \]

\[ T_{ij} = \lambda \delta_{ij} + 2\mu E_{ij} \]

\[ F_x = \int_v \rho \frac{\partial^2 u}{\partial t^2} \, dx \, dy \, dz = \int \int \int \left[ (\mu + \lambda) \frac{\partial}{\partial x} (\nabla \cdot u) + \mu \nabla^2 u_x \right] \, dx \, dy \, dz \]

Finally, 3-D wave equation:

\[ \rho \frac{\partial^2 u}{\partial t^2}(r, t) = (\mu + \lambda) \nabla [(\nabla \cdot u(r, t)] + \mu \nabla^2 u(r, t) \]
Dynamics of 3-D Continuum

Fourier Transform of 3-D Wave Equation

\[ \rho \frac{\partial^2 u}{\partial t^2}(r, t) = (\mu + \lambda) \nabla [(\nabla \cdot u(r, t)] + \mu \nabla^2 u(r, t) \]

Anticipating plane wave solutions, we Fourier Transform the equation:

\[ u(r, t) = \int \frac{d\omega}{2\pi} \int \frac{d^3 q}{(2\pi)^3} U(q, \omega) e^{i(q \cdot r - \omega t)} \]

\[ \rho \omega^2 U(q, \omega) = (\lambda + \mu) q [q \cdot U(q, \omega)] + \mu q^2 U(q, \omega) \]

Three coupled equations for \( U_x, U_y, \) and \( U_z \)....
Dynamics of 3-D Continuum

Dynamical Matrix

\[ \rho \omega^2 U_i(q, \omega) = (\lambda + \mu)q_i [q \cdot U(q, \omega)] + \mu q^2 U_i(q, \omega) \]

Express the system of equations as a matrix….

\[
\begin{bmatrix}
\rho \omega^2 & U_1 \\
\rho \omega^2 & U_2 \\
\rho \omega^2 & U_3
\end{bmatrix}
= 
\begin{bmatrix}
\mu q^2 + (\lambda + \mu)q_1^2 & (\lambda + \mu)q_1 q_2 & (\lambda + \mu)q_1 q_3 \\
(\lambda + \mu)q_2 q_1 & \mu q^2 + (\lambda + \mu)q_2^2 & (\lambda + \mu)q_2 q_3 \\
(\lambda + \mu)q_3 q_1 & (\lambda + \mu)q_3 q_2 & \mu q^2 + (\lambda + \mu)q_3^2
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3
\end{bmatrix}
\]

Turns the problem into an eigenvalue problem for the polarizations of the modes (eigenvectors) and wavevectors \( q \) (eigenvalues)….

\[ \rho \omega^2 U = D U \]
Dynamics of 3-D Continuum
Solutions to 3-D Wave Equation

\[ \rho \omega^2 U_i(q, \omega) = (\lambda + \mu)q_i [q \cdot U(q, \omega)] + \mu q^2 U_i(q, \omega) \]

Transverse polarization waves:

\[ q \cdot U(q, \omega) = 0 \]

\[ \rho \omega^2 = \mu q^2 \quad \text{for transverse waves} \]

\[ \omega = c_T |q| \quad \text{where} \quad c_T = \sqrt{\frac{\mu}{\rho}} \]

Longitudinal polarization waves:

\[ q \cdot U(q, \omega) = q U \]

\[ \rho \omega^2 U = (\lambda + 2\mu)q^2 U \quad \text{for longitudinal waves} \]

\[ \omega = c_L |q| \quad \text{where} \quad c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \]
Dynamics of 3-D Continuum

Summary

1. Dynamical Equation can be solved by inspection

\[ \rho \omega^2 U(q, \omega) = (\lambda + \mu) q [q \cdot U(q, \omega)] + \mu q^2 U(q, \omega) \]

2. There are 2 transverse and 1 longitudinal polarizations for each \( q \)

3. The dispersion relations are linear

\[ \omega = c_i |q| \]

\[ c_T = \sqrt{\frac{\mu}{\rho}} \quad c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \]

4. The longitudinal sound velocity is always greater than the transverse sound velocity

\[ \frac{c_L}{c_T} = \left( \frac{\lambda + 2\mu}{\mu} \right)^{1/2} = \left( 1 + \frac{1}{1 - 2\nu} \right)^{1/2} \]