Lecture 16: Type II Superconductors

Outline

1. A Superconducting Vortex

2. Vortex Fields and Currents

3. General Thermodynamic Concepts
   • First and Second Law
   • Entropy
   • Gibbs Free Energy (and co-energy)

4. Equilibrium Phase diagrams

5. Critical Fields

November 3, 2005

Fluxoid Quantization and Type II Superconductors

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The Vortex State

\[ \langle B \rangle = n_V \Phi_V \]

\( n_V \) is the areal density of vortices, the number per unit area.

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Please see: "A current-carrying type II superconductor in the mixed state" from http://phys.kent.edu/pages/cep.htm

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Quantized Vortices

Fluxoid Quantization along \( C_1 \)

\[ n\Phi_0 = \oint_{C_1} \mu_0 \lambda^2 \mathbf{J} \cdot d\mathbf{l} + \int_{S_1} \mathbf{B} \cdot d\mathbf{s} \]

But along the hexagonal path \( C_1 \), \( \mathbf{B} \) is a minimum, so that \( \mathbf{J} \) vanishes along this path.

Therefore, \( n\Phi_0 = \int_{S_1} \mathbf{B} \cdot d\mathbf{s} \)

And experiments give \( n = 1 \), so each vortex has one flux quantum associated with it.

For small \( C_2 \), \( \Phi_0 = \lim_{r \to 0} \oint_{C_2} \mu_0 \lambda^2 \mathbf{J} \cdot d\mathbf{l} \)

\[ \lim_{r \to 0} \mathbf{J} = \frac{\Phi_0}{2\pi \mu_0 \lambda^2} \]

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Normal Core of the Vortex

The current density \( \lim_{r \to 0} J_s = \frac{\Phi_o}{2\pi \mu_0 \lambda^2} \frac{1}{r} i_\phi \) diverges near the vortex center.

Which would mean that the kinetic energy of the superelectrons would also diverge. So to prevent this, below some core radius \( \xi \) the electrons become normal. This happens when the increase in kinetic energy is of the order of the gap energy. The maximum current density is then

\[
J_s^{\text{max}} = \frac{\Phi_o}{2\pi \mu_0 \lambda^2} \frac{1}{\xi} i_\phi \quad \Rightarrow \quad v_s^{\text{max}} = \frac{\hbar}{m^*} \frac{1}{\xi} i_\phi
\]

In the absence of any current flux, the superelectrons have zero net velocity but have a speed of the fermi velocity, \( v_F \). Hence the kinetic energy with currents is

\[
E_{\text{kin}}^0 = \frac{1}{2} m^* v_F^2 = \frac{1}{2} m^* \left( v_{F,x}^2 + v_{F,y}^2 + v_{F,z}^2 \right)
\]

The full BCS theory gives the coherence length \( \xi \)

\[\xi = \frac{\hbar v_F}{\pi \Delta_n} \]

Therefore the maximum current density, known as the depairing current density, is

\[J_{\text{depair}} \approx \frac{\Phi_o}{2\pi \mu_0 \lambda^2 \xi}\]

Coherence Length \( \xi \)

The energy of a superelectron at the core is

\[
E_{\text{kin}}^1 = \frac{1}{2} m^* \left( v_{F,x}^2 + v_{F,y}^2 + v_{s,\phi}^{\text{max}} \right)^2 + v_{F,z}^2
\]

The difference in energy, is to first order in the change in velocity,

\[\delta E \approx m^* v_{F,y} v_{s,\phi}^{\text{max}} \approx \Delta \]

With \( v_{s,\phi}^{\text{max}} = \frac{\hbar}{m^*} \frac{1}{\xi} i_\phi \) this gives \( \xi \approx \frac{\hbar v_F}{\Delta} \)

The full BCS theory gives the coherence length as \( \xi_o = \frac{\hbar v_F}{\pi \Delta_n} \)

Therefore the maximum current density, known as the depairing current density, is

\[J_{\text{depair}} \approx \frac{\Phi_o}{2\pi \mu_0 \lambda^2 \xi}\]
Temperature Dependence

Both the coherence length and the penetration depth diverge at $T_c$,

$$\lim_{T \to T_c} \xi(T) = \frac{\xi(0)}{\sqrt{1 - (T/T_c)}} \quad \text{and} \quad \lim_{T \to T_c} \lambda(T) = \frac{\lambda(0)}{\sqrt{1 - (T/T_c)}}$$

But there ratio, the Ginzburg-Landau parameter is independent of temperature near $T_c$,

$$\kappa = \frac{\lambda}{\xi}$$

$\kappa < 1/\sqrt{2}$ \quad Type I superconductor \quad Al, Nb

$\kappa > 1/\sqrt{2}$ \quad Type II superconductor \quad Nb, Most magnet materials \quad $\kappa \gg 1$

Vortex in a Cylinder

London’s Equations hold in the superconductor

$$\nabla \times (\Lambda J_s) = -\mathbf{B}$$

With Ampere’s Law gives

$$\nabla^2 \mathbf{B}(r) - \frac{1}{\lambda^2} \mathbf{B}(r) = 0 \quad \text{for} \quad r \geq \xi$$

Because $\mathbf{B}$ is in the $z$-direction, this becomes a scalar Helmholtz Equation

$$\nabla^2 B_z - \frac{1}{\lambda^2} B_z = 0 \quad \text{for} \quad r \geq \xi$$

Vortex in a cylinder

Which as a solution for an azimuthally symmetric field

\[
B_z(r) = \begin{cases} 
C_0 K_0 \left( \frac{r}{\lambda} \right) & \text{for } r \geq \xi \\
C_0 K_0 \left( \frac{\xi}{\lambda} \right) & \text{for } r < \xi 
\end{cases}
\]

C_0 is found from flux quantization around the core,

\[
C_0 = \frac{\Phi_o}{2\pi\lambda^2} \left[ \frac{1}{2} \frac{\xi^2}{\lambda^2} K_0 \left( \frac{\xi}{\lambda} \right) + \frac{\xi}{\lambda} K_1 \left( \frac{\xi}{\lambda} \right) \right]^{-1}
\]

Which for \( \kappa \gg 1 \)

\[
C_0 = \frac{\Phi_o}{2\pi\lambda^2}
\]

Vortex in a cylinder \( \kappa \gg 1 \)

\[
B(r) = \begin{cases} 
\frac{\Phi_o}{2\pi\lambda^2} K_0 \left( \frac{r}{\lambda} \right) i_z & \text{for } r \geq \xi \\
\frac{\Phi_o}{2\pi\lambda^2} K_0 \left( \frac{\xi}{\lambda} \right) i_z & \text{for } r < \xi 
\end{cases}
\]

\[
J_s(r) = \begin{cases} 
\frac{\Phi_o}{2\pi \mu_o \lambda^3} K_1 \left( \frac{r}{\lambda} \right) i_\phi & \text{for } r \geq \xi \\
0 & \text{for } r < \xi 
\end{cases}
\]
Energy of a single Vortex

The Electromagnetic energy in the superconducting region for a vortex is

\[ W_s = \frac{1}{2\mu_0} \int_{V_s} \left[ B^2 + \mu_0 J_s \cdot (\Lambda J_s) \right] dv \]

This gives the energy per unit length of the vortex as

\[ \mathcal{E}_V = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} K_0 \left( \frac{\xi}{\lambda} \right) \]

In the high \( \kappa \) limit this is

\[ \lim_{\lambda \gg \xi} \mathcal{E}_V = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} \ln \left( \frac{\lambda}{\xi} \right) \]

Modified London Equation \( \kappa >> \lambda/\xi \)

Given that one is most concerned with the high \( \kappa \) limit, one approximates the core of the vortex \( \xi \) as a delta function which satisfies the fluxoid quantization condition. This is known as the Modified London Equation:

\[ \nabla \times (\Lambda J_s) + B = V(r) \]

The vorticity is given by delta function along the direction of the core of the vortex and the strength of the vortex is \( \Phi_0 \)

For a single vortex along the z-axis: For multiple vortices

\[ V(r) = \Phi_0 \delta_2(r) i_z \quad V(r) = \sum_p \Phi_0 \delta_2(r - r_p) i_z \]
General Thermodynamic Concepts

First Law of Thermodynamics: conservation of energy

\[ dU = dQ + dW - \int \eta d\eta \]

- \( dU \): Internal energy
- \( dQ \): Heat in
- \( dW \): E&M energy stored
- \( \int \eta d\eta \): work done by the system

W: Electromagnetic Energy

Normal region of Volume \( V_n \)

\[ W_n = \int_{V_n} \frac{1}{2\mu_0} B^2 \, dv \]

Superconducting region of Volume \( V_s \)

\[ W_s = \frac{1}{2\mu_0} \int_{V_s} \left( B^2 + \mu_0 \mathbf{J}_S \cdot (\nabla \mathbf{J}_S) \right) \, dv \]

In the absence of applied currents, in Method II, we have found that

\[ dW = \int_V \mathbf{H} \cdot d\mathbf{B} \, dv \]

Moreover, for the simple geometries \( \mathbf{H} \) is a constant, proportional to the applied field. For a \( \mathbf{H} \) along a cylinder or for a slab, \( \mathbf{H} \) is just the applied field. Therefore,

\[ dW = \mathbf{H} \cdot d \int_V \mathbf{B} \, dv \]
Thermodynamic Fields

\[ dW = \mathbf{H} \cdot d \int_V \mathbf{B} \, dv \]

\[ \mathbf{\mathcal{H}} \equiv \mathbf{H} \quad \text{thermodynamic magnetic field} \]

\[ \mathbf{\mathcal{B}} = \frac{1}{V} \int_V \mathbf{B} \quad \text{thermodynamic flux density} \]

\[ \mathbf{\mathcal{M}} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{\mathcal{H}} \quad \text{thermodynamic magnetization density} \]

Therefore, the thermodynamic energy stored can be written simply as

\[ dW = V \mathbf{\mathcal{H}} \cdot d\mathbf{B} \]

Entropy and the Second Law

The entropy \( S \) is defined in terms of the heat delivered to a system at a temperature \( T \)

\[ dS \equiv \frac{dQ}{T} \]

Second Law of Thermodynamics:

For an isolated system in equilibrium \( \Delta S = 0 \)

The first law for thermodynamics for a system in equilibrium can be written as

\[ dU = T \, dS + V \mathbf{\mathcal{H}} \cdot d\mathbf{B} - f_\eta (d\eta) \]

Then the internal energy is a function of \( S, B, \) and \( \eta \)

\[ U = U(S, B, \eta) \]

\[ T, \mathcal{H}, f_\eta \quad \text{Conjugate variables} \]
Concept of Reservoir and Subsystem

Because we have more control over the conjugate variables $T, \mathcal{H}, f_i$, we seek a rewrite the thermodynamics in terms of these controllable variables.

Isolated system = Subsystem + Reservoir

$$\Delta S_{\text{tot}} = \Delta S_A + \Delta S_R$$

The change in entropy of the reservoir is

$$\Delta S_R = \frac{\Delta Q_R}{T_R} = -\frac{\Delta Q_A}{T_R}$$

Therefore,

$$\Delta S_{\text{tot}} = \frac{T_R \Delta S_A - \Delta Q_A}{T_R}$$

Gibbs Free Energy

The change total entropy is then

$$\Delta S_{\text{tot}} = -\frac{\Delta G_A - f_\eta \Delta \eta}{T_R} \geq 0$$

where the Gibbs Free Energy is defined by

$$G_A \equiv -T_R S_A + U_A + \frac{V}{T_R} \cdot \Delta \mathcal{H} - f_\eta \Delta \eta$$

At equilibrium, the available work is just $\Delta G$ (the energy that can be freed up to do work) and the force is decreasing:

$$f_\eta = -\left. \frac{\partial G}{\partial \eta} \right|_{T, \mathcal{H}}$$

$$\Delta G \leq 0$$
Gibbs Free Energy and Co-energy

The Gibbs free energy is

\[ G = -TS + U - V\hat{H} \cdot \vec{B} \]

The differential of \( G \) is

\[ dG = -TdS - SdT + dU - V\hat{H} \cdot d\vec{B} - V\hat{H} \cdot d\vec{B} \]

and with the use of the first law

\[ dU = TdS + V\hat{H} \cdot d\vec{B} - f_\eta d\eta \]

Therefore, the Gibbs free energy is a function of \( T, \hat{H}, \eta \).

At constant temperature and no work, then

\[ dG|_{T,\hat{H}} = -d\bar{W} \]

the co-energy

\[ f_\eta = -\left. \frac{\partial G}{\partial \eta} \right|_{T,\hat{H}} = \left. \frac{\partial \bar{W}}{\partial \eta} \right|_{T,\hat{H}} \]

Note minus sign!

Gibbs Free Energy and Equilibrium

In Equilibrium \( \Delta G = 0 \)

Consider the system made up of two phases 1 and 2.

Phase 1, \( G = G_1 \)  
Phase 2, \( G = G_2 \)  
Mixed phase \( G_{\text{tot}} = G_1 \frac{V_1}{V} + G_2 \frac{V_2}{V} \)

Therefore, \( G_{\text{tot}} = (G_1 - G_2) \frac{V_1}{V} + G_2 \) is minimized when \( G_1 = G_2 \)

Two phases in equilibrium with each other have the same Gibbs Free Energy
Phase Diagram and Critical Field

ΔG < 0 So that G is always minimized, the system goes to the state of lowest Gibbs Free Energy. At the phase boundary, \( G_s = G_n \).

At zero magnetic field in the superconducting phase

\[
G_s(\vec{H}, 0) < G_n(\vec{H}, 0)
\]

for \( T < T_c \)

\[
G_s(0, T) - G_n(0, T) = -\frac{1}{2} \mu_0 H_c^2(T) V_s
\]

condensation energy

The Thermodynamic Critical Field \( H_c(T) \) is experimentally of the form

\[
H_c(T) \approx H_{co} \left( 1 - \left( \frac{T}{T_c} \right)^2 \right)
\]

for \( T \leq T_c \)

Critical Field for Type I

Recall that

\[
dG = -V \vec{B} \cdot d\vec{H}
\]

In the bulk limit in the superconducting state \( B = 0 \) so that \( dG_s = 0 \)

Likewise in the normal state \( \vec{H} = H_{apf} \) and \( \vec{B} = \mu_0 \vec{H} \) so that

\[
dG_n = -V \mu_0 \vec{H} \cdot d\vec{H}
\]

Hence, we can write

\[
d \left( G_s(\vec{H}, T) - G_n(\vec{H}, T) \right) = V \mu_0 \vec{H} \cdot d\vec{H}
\]

Integration of the field from 0 to \( H \) gives

\[
G_s(\vec{H}, T) - G_n(\vec{H}, T) = G_s(0, T) - G_n(0, T) + \frac{1}{2} V \mu_0 \vec{H}^2
\]

and thus

\[
G_s(\vec{H}, T) - G_n(\vec{H}, T) = \frac{1}{2} \mu_0 \left( \vec{H}^2 - H_c^2 \right) V
\]
Critical Fields for Type II

The lower critical field $H_{c1}$ is the phase boundary where equilibrium between having one vortex and no vortex in the superconducting state.

$G_2^2(T, H) = G_2^0(T) + W_0 - \mu \cdot \int_{V_s} B \, dV$

Therefore

$H_{c1} = \frac{\mu}{\Phi_0} \ln \frac{\lambda}{\xi}$

The upper critical field $H_{c2}$ occurs when the flux density is such that the cores overlap:

$H_{c2} = \frac{\Phi_0}{2\pi \mu \xi^2}$