Lecture 7: Transmission Lines

Outline

1. Ladder Network Approximation
2. Inductance
3. Superconducting Transmission Line
4. Comparison with normal transmission line

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Transmission Line: circuit model

\[ \hat{v}(x) - \hat{v}(x + \Delta x) = (j \omega L_0 + R_0) \Delta x \hat{i}(x) \]

\[ \hat{i}(x) - \hat{i}(x + \Delta x) = j \omega C_0 \Delta x \hat{v}(x + \Delta x) \]
Transmission Line

\[
\frac{d\hat{v}}{dx} = - (j\omega L_0 + R_0) \hat{i} \quad \frac{d\hat{n}}{dx} = -j\omega C_0 \hat{v}
\]

A wave equation is obtained

\[
\frac{d^2\hat{v}}{dx^2} = - \left( \omega^2 L_0 C_0 - j\omega R_0 C_0 \right) \hat{v}
\]

Which has solutions of the form \( \hat{v}(x) = \hat{V} e^{-j k_0 x} \) with

\[
k_0 = \omega \sqrt{L_0 C_0} \sqrt{1 - j \left( R_0 / \omega L_0 \right)}
\]

Transmission line parameters

In the limit where the inductive impedance dominates,

\[
\lim_{\omega \tau_{LR} \gg 1} k_0 = \omega \sqrt{L_0 C_0} - j \frac{1}{2} \frac{R_0}{\sqrt{L_0 C_0}}
\]

So that \( v(x, t) = \text{Re} \left\{ \hat{V} e^{-\alpha x} e^{j\omega \left( t - \left( x / v_p \right) \right)} \right\} \)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_p )</td>
<td>Phase Velocity</td>
<td>( \frac{1}{\sqrt{L_0 C_0}} )</td>
</tr>
<tr>
<td>( 2\alpha )</td>
<td>Power Attenuation per Unit Length</td>
<td>( \frac{R_0}{\sqrt{L_0 C_0}} )</td>
</tr>
<tr>
<td>( Z_0 )</td>
<td>Characteristic Impedance</td>
<td>( \sqrt{L_0 \left( 1 - j \frac{1}{2} \frac{R_0}{2\omega L_0} \right)} )</td>
</tr>
</tbody>
</table>
Fields in the Transmission Line

\[ \hat{E}_y = \frac{\tau}{d} \sinh \left( \frac{b - z + h/2}{\lambda} \right) \sinh b/\lambda \quad \text{for} \quad 0 \leq z - (h/2) \leq b \]

\[ \hat{H}_y = \frac{\tau}{d} \quad \text{for} \quad |z| \leq h/2 \]

\[ \hat{E}_y = \frac{\tau}{d} \sinh \left( \frac{b + z + h/2}{\lambda} \right) \sinh b/\lambda \quad \text{for} \quad -b \leq z + (h/2) \leq C \]

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Fields in the Transmission Line

\[ \hat{E}_y = \frac{\tau}{d} \sinh \left( \frac{b_1 - z + h/2}{\lambda_1} \right) \sinh b_1/\lambda_1 \quad \text{for} \quad 0 \leq z - (h/2) \leq b_1 \]

\[ \hat{H}_y = \frac{\tau}{d} \quad \text{for} \quad |z| \leq h/2 \]

\[ \hat{E}_y = \frac{\tau}{d} \sinh \left( \frac{b_2 + z + h/2}{\lambda_2} \right) \sinh b_2/\lambda_2 \quad \text{for} \quad -b \leq z + (h/2) \leq C \]
Bulk Superconducting Transmission Line

\[ C_o = \epsilon_t \frac{d}{h} \]

\[
\lim_{\lambda \ll \delta} \lim_{\lambda \ll \delta} \frac{R_o}{\theta_0} = \frac{4}{d \theta_0 \rho_0} \left( \frac{\lambda}{\delta} \right)^3 = \frac{2}{d} \text{Re} \{ Z_S \}
\]

\[
\lim_{\lambda \ll \delta} \lim_{\lambda \ll \delta} L_o = \mu_0 \frac{h}{d} + 2\mu_0 \frac{\lambda}{d}
\]

\[
\lim_{\lambda \ll \delta} \lim_{\lambda \ll \delta} L_o = \frac{2}{d} \text{Im} \{ Z_S \}
\]

Inductance

Inductance per unit length is found from

\[
\frac{1}{4} \int dy \int dz \left( \mu |\vec{H}|^2 + \Lambda |\vec{J}_s|^2 \right) = \frac{1}{4} L_o |\vec{H}|^2
\]

Inside the transmission line space

\[ \vec{H}_y = \frac{\vec{H}}{d} \quad \text{for} \quad |z| \leq h/2 \]

\[ L_{o,\text{in}} = \mu \left( \frac{1}{d} \right)^2 \frac{h}{d} \]

Inside the transmission line material

\[ \vec{H}_y = \frac{\frac{\vec{H}}{d}}{\frac{\sinh k(b - z + (h/2))}{\sinh kb}} \quad \text{for} \quad 0 \leq z - (h/2) \leq b \]

\[
\lim_{\lambda \ll \delta} \lim_{\lambda \ll \delta} L_o,\text{material} = 2L_s = 2\mu_0 \frac{\lambda}{d}
\]
Inductance for a thin slab

The current density is uniform for then slab so that

$$J = \frac{\hat{I}}{bd} i_x$$

The energy stored in the slab is

$$W = \frac{1}{2} \mu_0 \lambda^2 (J)^2 b = \mu_0 \lambda^2 \left( \frac{\hat{I}}{bd} \right)^2 bd \Delta x = \frac{1}{2} L_o \Delta x \hat{I}^2$$

Therefore, $$L_o = \frac{\mu_0 \lambda^2}{bd}$$ For each slab and the total inductance per unit length is twice this. This is the kinetic inductance.

Dispersionless Transmission Lines

Because $L_o$ and $C_o$ do not depend on frequency for a superconductor, the phase velocity is independent of frequency. So that a pulse will propagate down a superconducting transmission line without dispersing. Also, the amount of attenuation is extremely small, since this is due to $R_o$.

For a normal metal, $L_o$ depends on frequency so that there is dispersion, in addition to a much greater loss.
Table removed for copyright reasons.
Please see: Table 4.5 (whole table), page 171, from Orlando, T., and K. Delin.