High Speed Communication Circuits
Lecture 4
S-Parameters and Impedance Transformers

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What Happens When the Wave Hits a Boundary?

- Reflections can occur
What Happens When the Wave Hits a Boundary?

- At boundary
  - Orientation of H-field flips with respect to E-field
  - Current reverses direction with respect to voltage
What Happens At The Load Location?

- Voltage and currents at load are ratioed according to the load impedance

![Diagram showing incident and reflected waves at a load location.](image)

**Voltage at Load**

\[ V_i + V_r \]

**Current at Load**

\[ I_i - I_r \]

**Ratio at Load**

\[ \frac{V_i + V_r}{I_i - I_r} = Z_L \]
Relate to Characteristic Impedance

- From previous slide

\[ \frac{V_i + V_r}{I_i - I_r} = \frac{V_i}{I_i} \left( \frac{1 + V_r/V_i}{1 - I_r/I_i} \right) = Z_L \]

- Voltage and current ratio in transmission line set by it characteristic impedance

\[ \frac{V_i}{I_i} = \frac{V_r}{I_r} = Z_o \quad \Rightarrow \quad \frac{I_r}{I_i} = \frac{V_r}{V_i} \]

- Substituting:

\[ Z_o \left( \frac{1 + V_r/V_i}{1 - V_r/V_i} \right) = Z_L \]
Define Reflection Coefficient

Definition: \[ \Gamma_L = \frac{V_r}{V_i} \]

- No reflection if \( \Gamma_L = 0 \)

Relation to load and characteristic impedances

\[ Z_o \left( \frac{1 + \Gamma_L}{1 - \Gamma_L} \right) = Z_L \]

Alternate expression

\[ \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} \]

- No reflection if \( Z_L = Z_o \)
Circuits or passive structures are often connected to transmission lines at high frequencies.

- How do you describe their behavior?
Calculate Response to Input Voltage Sources

- Assume source impedances match their respective transmission lines
Calculate Response to Input Voltage Sources

- Sources create incident waves on their respective transmission line
- Circuit/passive network causes
  - Reflections on same transmission line
  - Feedthrough to other transmission line
Calculate Response to Input Voltage Sources

- Reflections on same transmission line are parameterized by $\Gamma_L$
  - Note that $\Gamma_L$ is generally different on each side of the circuit/passive network

How do we parameterize feedthrough to the other transmission line?
S-Parameters – Definition

- Model circuit/passive network using 2-port techniques
  - Similar idea to Thevenin/Norton modeling

- Defining equations:

\[
\frac{V_{r1}}{\sqrt{Z_1}} = S_{11} \frac{V_{i1}}{\sqrt{Z_1}} + S_{12} \frac{V_{i2}}{\sqrt{Z_2}} \\
\frac{V_{r2}}{\sqrt{Z_2}} = S_{21} \frac{V_{i1}}{\sqrt{Z_1}} + S_{22} \frac{V_{i2}}{\sqrt{Z_2}}
\]
S-Parameters – Calculation/Measurement

\[ \frac{V_{r1}}{\sqrt{Z_1}} = S_{11} \frac{V_{i1}}{\sqrt{Z_1}} + S_{12} \frac{V_{i2}}{\sqrt{Z_2}} \]

\[ \frac{V_{r2}}{\sqrt{Z_2}} = S_{21} \frac{V_{i1}}{\sqrt{Z_1}} + S_{22} \frac{V_{i2}}{\sqrt{Z_2}} \]

- Set \( V_{in2} = 0 \)
  \[ \Rightarrow S_{11} = \frac{V_{r1}}{V_{i1}} = \Gamma_{L1} \]
  \[ \Rightarrow S_{21} = \sqrt{\frac{Z_1}{Z_2}} \left( \frac{V_{r2}}{V_{i2}} \right) \]

- Set \( V_{in1} = 0 \)
  \[ \Rightarrow S_{22} = \frac{V_{r2}}{V_{i2}} = \Gamma_{L2} \]
  \[ \Rightarrow S_{12} = \sqrt{\frac{Z_2}{Z_1}} \left( \frac{V_{r1}}{V_{i2}} \right) \]
Note: Alternate Form for $S_{21}$ and $S_{12}$

$$V_{i1} = \frac{V_{in1}}{2}$$

$$V_{i2} = \frac{V_{in2}}{2}$$

$$\Gamma_{L1}$$

$$\Gamma_{L2}$$

$$V_{in1}$$

$$V_{in2}$$

$$Z_1$$

$$Z_2$$

$$V_{r1}$$

$$V_{r2}$$

Linear Network

Set $V_{in2} = 0$

$$\Rightarrow S_{11} = \frac{V_{r1}}{V_{i1}} = \Gamma_{L1}$$

$$\Rightarrow S_{21} = 2\sqrt{\frac{Z_1}{Z_2}} \left( \frac{V_{r2}}{V_{in1}} \right)$$

Set $V_{in1} = 0$

$$\Rightarrow S_{22} = \frac{V_{r2}}{V_{i2}} = \Gamma_{L2}$$

$$\Rightarrow S_{12} = 2\sqrt{\frac{Z_2}{Z_1}} \left( \frac{V_{r1}}{V_{in2}} \right)$$

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Key issue – two-port is parameterized with respect to the left and right side load impedances (Z₁ and Z₂)
- Need to recalculate $S_{11}$, $S_{21}$, etc. if $Z_1$ or $Z_2$ changes
- Typical assumption is that $Z_1 = Z_2 = 50$ Ohms
S-Parameter Calculations – Example 1

- **Set $V_{i2} = 0$**

  \[
  V_{r1} = \Gamma_1 V_{i1} = \frac{Z_2 - Z_1}{Z_2 + Z_1} V_{i1}
  \]

  \[
  V_{r2} = V_{i1} + V_{r1} = (1 + \Gamma_1) V_{i1}
  \]

  \[
  \Rightarrow S_{11} = \Gamma_1
  \]

  \[
  \Rightarrow S_{21} = \sqrt{\frac{Z_1}{Z_2}} (1 + \Gamma_1)
  \]

- **Set $V_{i1} = 0$**

  \[
  V_{r2} = \Gamma_2 V_{i2} = \frac{Z_1 - Z_2}{Z_1 + Z_2} V_{i2}
  \]

  \[
  V_{r1} = V_{i2} + V_{r2} = (1 + \Gamma_2) V_{i2}
  \]

  \[
  \Rightarrow S_{22} = \Gamma_2
  \]

  \[
  \Rightarrow S_{12} = \sqrt{\frac{Z_2}{Z_1}} (1 + \Gamma_2)
  \]
**S-Parameter Calculations – Example 2**

- **Same as before:**
  
  \[ S_{11} = \Gamma_1 \]
  
  \[ S_{21} = \sqrt{\frac{Z_1}{Z_2}} (1 + \Gamma_1) \]
  
  \[ S_{22} = \Gamma_2 \]
  
  \[ S_{12} = \sqrt{\frac{Z_2}{Z_1}} (1 + \Gamma_2) \]

- **But now:**
  
  \[ \Gamma_1 = \frac{Z_2 \| (1/sC') - Z_1}{Z_2 \| (1/sC') + Z_1} \]
  
  \[ \Gamma_2 = \frac{Z_1 \| (1/sC') - Z_2}{Z_1 \| (1/sC') + Z_2} \]
The S-parameter calculations are now more involved
- Network now has more than one node
- This is a homework problem
Impedance Transformers
Matching for Voltage versus Power Transfer

- Consider the voltage divider network
  Given the Thevenin equivalent source with $V_s$ and $R_s$, how do we deliver maximum voltage or power to the load?

- For maximum voltage transfer
  
  $$R_L \rightarrow \infty \implies V_{out} \rightarrow V_s$$

- For maximum power transfer
  
  $$R_L = R_S \implies P_{out} = \frac{|V_{out}|^2}{R_L} = \frac{|V_s|^2}{4R_S}$$

Which one do we want?
**Note: Maximum Power Transfer Derivation**

- **Formulation:** $R_s$ is given, $R_L$ is variable

\[
P_{out} = I^2 R_L = \left( \frac{V_s}{R_s + R_L} \right)^2 R_L = \frac{R_L}{(R_s + R_L)^2} V_s^2
\]

- **Take the derivative and set it to zero**

\[
\frac{dP_{out}}{dR_L} = R_L (-2) (R_s + R_L)^{-3} + (R_s + R_L)^{-2} = 0
\]

\[
\Rightarrow 2R_L = R_s + R_L \Rightarrow R_L = R_s
\]
Voltage Versus Power

- For most communication circuits, voltage (or current) is the key signal for detection
  - Phase information is important
  - Power is ambiguous with respect to phase information
    - Example:

- For high speed circuits with transmission lines, achieving maximum power transfer is important
  - Maximum power transfer coincides with having zero reflections (i.e., $\Gamma_L = 0$)

Can we ever win on both issues?
Broadband Impedance Transformers

- Consider placing an ideal transformer between source and load

![Transformer diagram](image)

- Transformer basics (passive, zero loss)

1) \( V_{out} = NV_{in} \)

2) Power In = Power Out

\[ \Rightarrow V_{in}I_{in} = V_{out}I_{out} \]

From (1) and (2): \( V_{in}I_{in} = NV_{in}I_{out} \) \( \Rightarrow I_{out} = \frac{I_{in}}{N} \)

- Transformer input impedance

\[ R_{in} = \frac{V_{in}}{I_{in}} = \frac{V_{out}/N}{NI_{out}} = \frac{1}{N^2}R_L \]
What Value of N Maximizes Voltage Transfer?

- Derive formula for $V_{out}$ versus $V_{in}$ for given N value

$$V_{out} = NV_{in} = N \frac{R_{in}}{R_s + R_{in}} V_s = N \frac{R_L/N^2}{R_s + R_L/N^2} V_s$$

$$= N \frac{R_L}{R_L + N^2 R_s} V_s$$

- Take the derivative and set it to zero

$$\frac{dV_{out}}{dN} = NR_L(-1)(R_L+N^2 R_s)^{-2} 2NR_s + R_L(R_L+N^2 R_s)^{-1} = 0$$

$$\Rightarrow -2N^2 R_s (R_L+N^2 R_s)^{-2} + (R_L+N^2 R_s)^{-1} = 0$$

$$\Rightarrow -2N^2 R_s = R_L + N^2 R_s \quad \Rightarrow N^2 = \frac{R_L}{R_s}$$
What is the Input Impedance for Max Voltage Transfer?

- We know from basic transformer theory that input impedance into transformer is

\[ R_{in} = \frac{1}{N^2} R_L \]

- We just learned that, to maximize voltage transfer, we must set the transformer turns ratio to

\[ N^2 = \frac{R_L}{R_s} \]

- Put them together

\[ R_{in} = \frac{1}{N^2} R_L = \frac{1}{R_L/R_s} R_L = R_s \]

So, N should be set for max power transfer into transformer to achieve the maximum voltage transfer at the load. This also ensures no reflection.
**Benefit of Impedance Matching with Transformers**

- Transformers allow maximum voltage and power transfer relationship to coincide

  ![Transformer Diagram]

- Turns ratio for max power/voltage transfer

  \[ N^2 = \frac{R_L}{R_S} \]

- Resulting voltage gain (can exceed one!)

  \[ V_{out} = NV_{in} = N \left( \frac{1}{2} V_s \right) = \sqrt{\frac{R_L}{R_S}} \left( \frac{1}{2} V_s \right) \]
Problems with True Transformers

- It’s difficult to realize a transformer with good performance over a wide frequency range
  - Magnetic materials have limited frequency response (both low and high frequency limits)
  - Inductors have self-resonant frequencies, losses, and mediocre coupling to other inductors without magnetic material

- For wireless applications, we only need transformers that operate over a small frequency range (except UWB)
  - Can we take advantage of this?: use ‘impedance transformer” instead of a true transformer
Consider Resonant Circuits (Chap. 3 (2nd ed.) or 4 (1st ed.) of Text)

Series Resonant Circuit

\[ Z_{in} = \frac{1}{jwC_s} + jwL_s + R_s \]

\[ = R_s \text{ for } w = \frac{1}{\sqrt{L_sC_s}} = w_o \]

\[ Q = \frac{w_oL_s}{R_s} = \frac{1}{w_oC_sR_s} \]

Parallel Resonant Circuit

\[ Z_{in} = \frac{1}{jwC_p} \parallel jwL_p \parallel R_p \]

\[ = R_p \text{ for } w = \frac{1}{\sqrt{L_pC_p}} = w_o \]

\[ Q = \frac{R_p}{w_oL_p} = w_oC_pR_p \]

- Key insight: at resonance \( Z_{in} \) becomes purely real despite the presence of reactive elements
Equivalece of Series and Parallel RL Circuits

- Equate real and imaginary parts of the left and right expressions (so that $Z_{in}$ is the same for both)
  - Also equate $Q$ values

\[ R_p = R_s (Q^2 + 1) \approx R_s Q^2 \] (for $Q \gg 1$)

\[ L_p = L_s \left( \frac{Q^2 + 1}{Q^2} \right) \approx L_s \] (for $Q \gg 1$)
Series-Parallel Equivalence Analysis

\[ Z_{in_p} = \frac{j\omega_o L_p R_p}{j\omega_o L_p R_p + R_p} \]

\[ = \frac{j\omega_o L_p R_p (-j\omega_o L_p R_p + R_p)}{\omega_o^2 L_p^2 + R_p^2} \]

\[ = \frac{\omega_o^2 L_p^2 R_p}{\omega_o^2 L_p^2 + R_p^2} \]

\[ = R_p \frac{1}{1 + Q_p^2} \]

\[ R_s = \frac{\omega_o^2 L_p^2 R_p}{\omega_o^2 L_p^2 + R_p^2} \]

\[ = \frac{L_p R_p^2}{\omega_o^2 L_p^2 + R_p^2} \]

\[ = L_p \frac{1}{\frac{\omega_o^2 L_p^2}{R_p^2} + 1} \]

\[ = L_p \frac{1}{1 + \frac{1}{Q_p^2}} \]

\[ Q_s = \frac{\omega_o L_s}{R_s} \]

\[ = \frac{\omega L_p Q_p^2}{R_p} \]

\[ = Q_p \]

\[ = Q \]
Equivilance of Series and Parallel RC Circuits

- Equate real and imaginary parts of the left and right expressions (so that $Z_{in}$ is the same for both)
- Also equate $Q$ values

$$R_p = R_s(Q^2 + 1) \approx R_s Q^2 \quad (\text{for } Q \gg 1)$$
$$C_p = C_s \left( \frac{Q^2}{Q^2 + 1} \right) \approx C_s \quad (\text{for } Q \gg 1)$$
A Narrowband Impedance Transformer: The L Match

At resonance

\[ Z_{in} = R_p = (1 + Q^2)R_s \approx Q^2R_s \quad \text{(purely real)} \]

Transformer steps up impedance!

\[ R_p = R_s(Q^2 + 1) \approx R_sQ^2 \quad \text{(for } Q \gg 1) \]

\[ L_p = L_s\left(\frac{Q^2 + 1}{Q^2}\right) \approx L_s \quad \text{(for } Q \gg 1) \]
Alternate Implementation of L Match

\[ \frac{R_p}{1 + Q^2} \approx \frac{R_p}{Q^2} \quad \text{(purely real)} \]

- **At resonance**
  \[ R_p = R_s(Q^2 + 1) \approx R_sQ^2 \quad \text{(for } Q \gg 1 \text{)} \]
  \[ C_p = C_s \left( \frac{Q^2}{Q^2 + 1} \right) \approx C_s \quad \text{(for } Q \gg 1 \text{)} \]

- **Transformer steps down impedance!**
The $\pi$ Match

- Combines two L sections

- Provides an extra degree of freedom for choosing component values for a desired transformation ratio

$$L_1 + L_2 = L$$

Steps Up Impedance

Steps Down Impedance
The T Match

- Also combines two L sections

- Again, benefit is in providing an extra degree of freedom in choosing component values
Tapped Capacitor as a Transformer

To first order:

\[ \frac{R_{in}}{R_L} \approx \left( \frac{C_1 + C_2}{C_1} \right)^2 \]

- Useful in VCO design
- See Chap. 3 (2nd ed.) or 4 (1st ed.) of Text