Fluids A

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*(with thanks to SDS)
Outline

> Intro to microfluidics

> Basic concepts: viscosity and surface tension

> Governing equations

> Incompressible laminar flow
Microfluidics

> The manipulation and use of fluids at the microscale

> Most fluid domains are in use at the microscale

- Explosive thermofluidic flows
  - Inkjet printheads
  - We will not cover this regime
- High-speed gas flows
  - Micro-turbomachinery
  - We will not cover this regime
- Low-speed gas flows
  - Squeeze-film damping
  - We’ll do a bit of this to get \( b \) for SMD
- Liquid-based slow flow
  - This will be the focus
Microfluidics

> This has been one of the most important domains of MEMS
  • Even though most microfluidics is not “MEMS”
  • And there are few commercial products

> The overall driver has been the life sciences
  • Though the only major commercial success is inkjets

> The initial driver was analytical chemistry
  • Separation of organic molecules

> More recently, this has shifted to biology
  • Manipulation of DNA, proteins, cells, tissues, etc.
Microfluidics examples

> **H-filter**

- Developed by Yager and colleagues at UWash in mid-90’s
- Being commercialized by Micronics

> **An intrinsically microscale device**

- Uses diffusion in laminar flow to separate molecules

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**Yager et al., Nature 2006**

Courtesy of Paul Yager, Thayne Edwards, Elain Fu, Kristen Helton, Kjell Nelson, Milton R. Tam, and Bernhard H. Weigl.

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Microfluidics

> Multi-layer elastomeric microfluidics (Quake, etc.)
  
  - Use low modulus of silicone elastomers to create hydraulic valves
  - Move liquids around
  - Use diffusivity of gas in elastomer to enable dead-end filling

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Fluidigm

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To design microfluidics we need to understand

- What pressures are needed for given flows
  » How do I size my channels?
- What can fluids do at these scales
  » What are the relevant physics?
- What things get better as we scale down
  » Mixing times, reagent volumes
- What things get worse, and how can we manage them
  » Surface tension
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Viscosity

> When a solid experiences shear stress, it deforms (e.g., strains)
  - Shear Modulus relates the two

> When a fluid experiences shear stress, it deforms continuously
  - Viscosity relates the two

> Constitutive property describing relationship between shear stress [Pa] and shear rate [s⁻¹]

> Units: Pa-s
  - Water: 0.001 Pa-s
  - Air: 10⁻⁵ Pa-s

\[
\tau = \eta \frac{U}{h}
\]

and, in the differential limit

\[
\tau = \eta \frac{\partial U_x}{\partial y}
\]

A related quantity: Kinematic Viscosity

\[
\eta^* = \frac{\eta}{\rho_m}
\]

This is a diffusivity for momentum

\[
\tau = \eta^* \frac{\partial (\rho_m U_x)}{\partial y}
\]

[m²/s]
Surface Tension

> A liquid drop minimizes its free energy by minimizing its surface area. The effective force responsible for this is called **surface tension** ($\Gamma$) [J/m$^2$ = N/m]

> The surface tension creates a differential pressure on the two sides of a curved liquid surface

$$ (2\pi r)\Gamma = \Delta P(\pi r^2) $$

solving for $\Delta P$

$$ \Delta P = \frac{2\Gamma}{r} $$
Capillary Effects

> Surface forces can actually transport liquids

> Contact angles determine what happens, and these depend on the wetting properties of the liquid and the solid surface.

\[ \rho_m g h \left(\pi r^2\right) = 2\pi r \Gamma \cos \theta \]

\[ h = \frac{2\Gamma \cos \theta}{\rho_m g r} \]

Image by MIT OpenCourseWare.
Capillary Effects

> A hydrophobic valve

Hydrophobic barrier

Fill

Stop

Burst

Image removed due to copyright restrictions.
Surface tension

> The scaling as 1/r is what makes dealing with surface tension HARD at the microscale

> Solutions

- Prime with low-surface tension liquids
  - Methanol (Γ=22.6 mN/m) vs. water (Γ=72.8 mN/m)
  - Or use surfactants
- Use CO₂ instead of air
  - Dissolves more readily in water
    - Zengerle et al., IEEE MEMS 1995, p340
- Use diffusivity of gas in PDMS
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Fluid Mechanics Governing Equations

> Mass is conserved $\rightarrow$ Continuity equation

> Momentum is conserved $\rightarrow$ Navier-Stokes equation

> Energy is conserved $\rightarrow$ Euler equation

> We consider only the first two in this lecture
Continuity equation

> Conservation of mass

> In this case, for a control or “fixed” fluid volume
  • both S and V are constant in time

> Point conservation relation is valid for fixed or moving point

\[
\frac{d}{dt} \int_{\text{volume}} b dV = - \int F \cdot n dS + \int g dV
\]

\[
\frac{\partial b}{\partial t} = -\nabla \cdot F + g
\]

Apply the divergence theorem:

\[
\int_{\text{volume}} \left[ \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m U) \right] dV = 0
\]

which implies

\[
\frac{\partial \rho_m}{\partial t} + U \cdot \nabla \rho_m + \rho_m \nabla \cdot U = 0
\]
Material Derivative

> The density can change due to three effects:
  • An explicit time dependence (e.g. local heating)
  • Flow carrying fluid through changing density regions
  • Divergence of the fluid velocity

> The first two of these are grouped into the “material derivative”
  • Rate of change for an observer moving with the fluid

Mathematically:

\[ \frac{\partial \rho_m}{\partial t} \]

\[ \mathbf{U} \cdot \nabla \rho_m \]

\[ \rho_m (\nabla \cdot \mathbf{U}) \]
Material Derivative

> The first two of these are grouped into the “material derivative”

\[
\frac{\partial \rho_m}{\partial t} + \mathbf{U} \cdot \nabla \rho_m + \rho_m \nabla \cdot \mathbf{U} = 0
\]

If we define

\[
\frac{D \rho_m}{Dt} = \frac{\partial \rho_m}{\partial t} + \mathbf{U} \cdot \nabla \rho_m
\]

or, more generally, define the operator

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla
\]

we can write the continuity equation as

\[
\frac{D \rho_m}{Dt} + \rho_m \nabla \cdot \mathbf{U} = 0
\]

If the density is uniform, then in steady state

\[
\nabla \cdot \mathbf{U} = 0
\]
Momentum Conservation

> We want to write Newton’s 2nd Law for fluids

\[
F = \frac{dp}{dt} = \frac{d(mU)}{dt}
\]

> We start with a volume that travels with the fluid

\( \Rightarrow \) contains a constant amount of mass

- Material volume

> Then go to an arbitrary control volume

- Must account for flux of momentum through surface

\[
F = \frac{d}{dt} \int_{V(t)} \rho_m UdV_m + \int_{S(t)} \rho_m U \left[ (U - U_s) \cdot n \right] dS
\]
Momentum Conservation

> Pull time derivative into integral

> Cancel terms

> Apply divergence theorem

\[
F = \frac{d}{dt} \int_{V(t)} \rho_m U dV + \int_{S(t)} \rho_m U [(U - U_s) \cdot n] dS
\]

= \int_{V(t)} \frac{\partial}{\partial t} \rho_m U dV + \int_{S(t)} \rho_m U [U_s \cdot n] dS + \int_{S(t)} \rho_m U [(U - U_s) \cdot n] dS

Leibniz’s rule

\[
= \int_{V(t)} \frac{\partial}{\partial t} \rho_m U dV + \int_{S(t)} \rho_m UU \cdot ndS
\]

\[\downarrow\int_{S(t)} a \cdot ndS = \int_{V(t)} \nabla \cdot adV\]
Momentum Conservation

> Final result does not depend on control volume

\[
\begin{align*}
    & = \int_{V(t)} \frac{\partial}{\partial t} \rho_m U dV + \int_{V(t)} \nabla \cdot \rho_m U U dV \\
    & = \int_{V(t)} \left[ U \left( \frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m U \right) + \rho_m \left( \frac{\partial U}{\partial t} + U \cdot \nabla U \right) \right] dV \\
    & \downarrow \\
    & F = \int_{V} \rho_m \frac{DU}{Dt} dV
\end{align*}
\]
Momentum Conservation

> Add in force terms

> Include body forces (e.g., gravity)

> And surface forces (i.e., stresses)

\[
F_b = \int_{V} \rho_m g \, dV
\]

\[
F_s = \int_{S(t)} s(n) \, dS
\]

\[
s(n) = n \cdot \sigma
\]

\[
F_s = \int_{S(t)} n \cdot \sigma dS
\]

\[
F_s = \int_{V(t)} \nabla \cdot \sigma \, dV
\]

\[
\rho_m \frac{D U}{D t} = \rho_m g + \nabla \cdot \sigma
\]
Navier-Stokes Equations

> Substitute in for stress tensor

\[ \sigma = -Pn + \tau \]
\[ \nabla \cdot \sigma = -\nabla P + \nabla \cdot \tau \]

> Compressible Newtonian fluid constitutive relation

\[ \nabla \cdot \tau \]
\[ \downarrow \]
\[ \tau = \eta \frac{\partial U_x}{\partial y} \]

> Compressible Navier-Stokes equations

\[ \eta \nabla^2 U + \frac{\eta}{3} \nabla (\nabla \cdot U) \]
\[ \downarrow \]

\[ P^* = P - \rho_m g \cdot r \]
\[ \rho_m \frac{DU}{Dt} = -\nabla P + \eta \nabla^2 U + \frac{\eta}{3} \nabla (\nabla \cdot U) + \rho_m g \]
\[ \rho_m \frac{DU}{Dt} = -\nabla P^* + \eta \nabla^2 U + \frac{\eta}{3} \nabla (\nabla \cdot U) \]
Navier-Stokes Equations

> Terms in compressible N-S equations

\[ \rho_m \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla P + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla (\nabla \cdot \mathbf{U}) + \rho_m \mathbf{g} \]

- time-dependence
- pressure
- compressibility
- inertial
- viscous stresses
- gravity
Dimensionless Numbers

> Fluid mechanics is full of non-dimensional numbers that help classify the types of flow

> Reynolds number is most important

> Reynolds number:
  • The ratio of inertial to viscous effects
  • Ratio of convective to diffusive momentum transport
  • Small Reynolds number means neglect of inertia
  • Flow at low Reynolds number is laminar

\[ \text{Re} = \frac{\rho m L_0 U_0}{\eta} = \frac{L_0 U_0}{\eta^*} = \frac{U_0}{\eta^* / L_0} \]

Non-dimensionalized steady incompressible flow

See Deen, Analysis of Transport Phenomena
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Incompressible Laminar Flow

> The Navier-Stokes equation becomes very “heat-flow-equation-like,” although the presence of $\frac{DU}{Dt}$ instead of $\frac{\partial U}{\partial t}$ makes the equation nonlinear, hence HARD.

Navier-Stokes becomes

$$\rho_m \frac{DU}{Dt} = -\nabla P^* + \eta \nabla^2 U$$

to obtain a "diffusion-like" equation:

$$\rho_m \frac{DU}{Dt} = \eta \nabla^2 U - \nabla P^*$$
As aside on heat convection

> Our heat-flow equation looked like

> Compare to incompressible N-S eqn

> If we allow fluid to move—to convect—we can include convection in our heat conservation

> At steady state, we get a relation that allows us to compare convective heat transport to conduction

> This is the Peclet number

> For microscale water flows, L~100 μm, U~0.1 mm/s, D~150×10^{-6} m^2/s

\[
\frac{\partial T}{\partial t} = D \nabla^2 T + \frac{1}{\rho_0} \frac{\partial P}{\partial t} \text{ sources}
\]

\[
\frac{\rho m}{D} \frac{DU}{Dt} = \eta \nabla^2 U - \nabla P^* 
\]

\[
\frac{DT}{Dt} = D \nabla^2 T + \frac{1}{\rho_0} \frac{\partial P}{\partial t} \text{ sources}
\]

\[
U \cdot \nabla T = D \nabla^2 T + \frac{1}{\rho_0} \frac{\partial P}{\partial t} \text{ sources}
\]

\[
P_e = \frac{LU}{D} = \frac{(10^{-4} m)(10^{-4} m/s)}{0.15 \cdot 10^{-6} m^2/s} \sim 0.1
\]
Couette or Shear Flow

- Pure shear flow with a linear velocity profile
- No pressure gradient
- Relative velocity goes to zero at the walls (the so-called no-slip boundary condition)

The flow is one-dimensional

$$U = U_x(y)\hat{n}_x$$

$$\rho_m \left( \frac{dU}{dt} + U \cdot \nabla U \right) = \eta \nabla^2 U - \nabla P^*$$

N-S Eqns collapse to the Laplace eqn

$$\frac{\partial^2 U_x}{\partial y^2} = 0$$

$$U_x = c_1 y + c_2$$

B.C.'s: $$U_x(0) = 0, U_x(h) = U$$

$$U_x = \frac{y}{h} U$$

Image by MIT OpenCourseWare.
Adapted from Figure 13.4 in Senturia, Stephen D. Microsystem Design.
Poiseuille Flow

> **Pressure-driven flow through a pipe**
  - In our case, two parallel plates

> **Velocity profile is parabolic**

> **This is the most common flow in microfluidics**
  - Assumes that $h << W$

Solution for Poiseuille Flow

- Assume a uniform pressure gradient along the pipe
- Assume x-velocity only depends on y
- Enforce zero-flow boundary conditions at walls
- Maximum velocity is at center
- Volumetric flowrate is

\[
\frac{dP}{dx} = -K
\]

\[
\frac{\partial^2 U_x}{\partial y^2} = -\frac{K}{\eta}
\]

\[
U_x = \frac{1}{2\eta} \left[ y(h - y) \right] K
\]

\[
U_{\text{max}} = \frac{h^2}{8\eta} K
\]

\[
Q = W \int_0^h U_x dy = \frac{Wh^3}{12\eta} K
\]
Lumped Model for Poiseuille Flow

> Can get lumped resistor using the fluidic convention

> Note STRONG dependence on $h$

> This relation is more complicated when the aspect ratio is not very high…

$$\Delta P = \text{effort} = KL$$

$$\Delta P = \frac{12\eta L}{Wh^3} Q$$

$$\Rightarrow R_{Pois} = \frac{12\eta L}{Wh^3}$$
Development Length

> It takes a certain characteristic length, called the **development length**, to establish the Poiseuille velocity profile.

> This development length corresponds to a development time for viscous stresses to diffuse from wall.

> Development length is proportional to the characteristic length scale and to the Reynolds number, both of which tend to be small in microfluidic devices.

\[
time \approx \frac{L^2}{\eta^*} \approx \text{Re} \frac{L}{U} \\
L_D \approx (\text{time})U \approx \text{Re} L
\]
A note on vorticity

> A common statement is to say that laminar flow has no vorticity

> What is meant is that laminar flow has no turbulence

> Vorticity and turbulence are different

> Can the pinwheel spin?
  • Then there is vorticity

> Demonstrate for Poiseuille flow

\[ U_x = \frac{1}{2\eta} [y(h - y)]K \]

\[ \omega = \nabla \times \mathbf{U} \]

\[ \omega = n_y \frac{\partial U_x}{\partial z} - n_z \frac{\partial U_x}{\partial y} \]

\[ \omega = -n_z \frac{K}{2\eta} (h - 2y) \]