MOVING AVERAGE CHART

- A goal is to detect small shifts as rapidly as possible, without increasing sample size.

- Approach: Combine results from multiple samples.

  Average of a moving window that contains \( w \) samples (each sample still of size \( n \))

  \[
  M_t = \frac{\bar{x}_t + \bar{x}_{t-1} + \ldots + \bar{x}_{t-w+1}}{w}
  \]

  \[
  \mu_{M} = \mu, \quad \sigma^2_{M} = \frac{\sigma^2}{nw}
  \]

  \[
  \begin{align*}
  UCL &= \bar{x} + \frac{3\sigma}{\sqrt{nw}} \\
  CL &= \bar{x} \\
  LCL &= \bar{x} - \frac{3\sigma}{\sqrt{nw}}
  \end{align*}
  \]

EXPONENTIALLY WEIGHTED MOVING AVERAGE (EWMA)

- If our goal is to catch shifts rapidly, we might consider weighing more recent samples more heavily than older samples:

  \[
  z_t = \lambda \bar{x}_t + (1-\lambda) z_{t-1}, \quad 0 < \lambda \leq 1, \quad z_0 = \bar{x}
  \]

  \[
  = \lambda \sum_{j=0}^{t-1} (1-\lambda)^j \bar{x}_{t-j} + (1-\lambda)^t z_0
  \]

  \[
  \begin{align*}
  UCL &= \bar{x} + 3 \sqrt{\frac{\lambda}{(2-\lambda)n}} \\
  LCL &= \bar{x} - 3 \sqrt{\frac{\lambda}{(2-\lambda)n}}
  \end{align*}
  \]

  \[
  \sigma^2_{z_t} = \frac{\sigma^2}{n} \frac{\lambda}{2-\lambda} (1 - (1-\lambda)^2t)
  \]

  In general (use for small \( t \))

- Smaller \( \lambda \) \( \Rightarrow \) to detect smaller shifts ("accumulate more history")
EWMA Chart Design

- Choices: (1) Other alternatives than $\pm 3 \sigma$
(2) EWMA filter coefficient $\lambda$

Approach: Choose based on ARL to detect shift of desired size

- Often used with individuals chart: utilize many past runs to detect shift.

- Typical values: $\lambda = 0.1$
$\pm 2.7 \hat{z}$

$$\text{ARL}_0 \geq 500 \quad \text{for false alarm}$$
$$\text{ARL}_1 \geq 10.3 \quad \text{for detecting a shift of } 1 \sigma$$

Similar to Shewhart/mrco:

$\lambda = 0.4$
$\pm 3.054 \hat{z}$

$$\text{ARL}_0 \geq 500 \quad \text{for false alarm}$$
$$\text{ARL}_1 \geq 14.3 \quad \text{for detecting a shift of } 1 \sigma$$

Average Run Lengths for Several EWMA Control Schemes [Adapted from Lucas and Saccucci (1990)]

<table>
<thead>
<tr>
<th>Shift or Mean (multiple of $1\sigma$)</th>
<th>$L = 3.054$</th>
<th>2.998</th>
<th>2.962</th>
<th>2.814</th>
<th>2.615</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 0.40$</td>
<td>500</td>
<td>500</td>
<td>500</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>0.25</td>
<td>224</td>
<td>170</td>
<td>150</td>
<td>106</td>
<td>84.1</td>
</tr>
<tr>
<td>0.50</td>
<td>71.2</td>
<td>48.2</td>
<td>41.8</td>
<td>31.3</td>
<td>28.8</td>
</tr>
<tr>
<td>0.75</td>
<td>28.4</td>
<td>20.1</td>
<td>18.2</td>
<td>15.9</td>
<td>16.4</td>
</tr>
<tr>
<td>1.00</td>
<td>14.3</td>
<td>11.1</td>
<td>10.5</td>
<td>10.3</td>
<td>11.4</td>
</tr>
<tr>
<td>1.50</td>
<td>5.9</td>
<td>5.5</td>
<td>5.5</td>
<td>6.1</td>
<td>7.1</td>
</tr>
<tr>
<td>2.00</td>
<td>3.5</td>
<td>3.6</td>
<td>3.7</td>
<td>4.4</td>
<td>5.2</td>
</tr>
<tr>
<td>2.50</td>
<td>2.5</td>
<td>2.7</td>
<td>2.9</td>
<td>3.4</td>
<td>4.2</td>
</tr>
<tr>
<td>3.00</td>
<td>2.0</td>
<td>2.3</td>
<td>2.4</td>
<td>2.9</td>
<td>3.5</td>
</tr>
<tr>
<td>4.00</td>
<td>1.4</td>
<td>1.7</td>
<td>1.9</td>
<td>2.2</td>
<td>2.7</td>
</tr>
</tbody>
</table>

- EWMA (exponentially weighted moving variance)
  extensions are also available
CUSUM CHARTS

Principle: Again, use multiple past history points to accelerate detection of a shift.

⇒ Plot cumulative sums of deviations of sample from a target value (with some slack.)

• Very useful for $n=1$ sampling

\[ C_i = \sum_{j=1}^{i} (x_i - \mu_0) \]
\[ C_i = (x_i - \mu_0) + C_{i-1} \]

• Control chart design approaches
  (1) "tabular" or "algorithmic" CUSUM
      + some modifications to summation
      + looks similar to normal chart with upper/lower control limits

(2) "V-mask" design
    + approach often seen
    - hard to read and interpret
Tabular Cusum

\[ C_{i+} = \max \{ 0, x_i - (\bar{\mu}_0 + k) + C_{i-1}^+ \} \]

\[ C_{i-} = \max \{ 0, (\bar{\mu}_0 - k) - x_i + C_{i-1}^- \} \]

with \( C_{0+} = C_{0-} = 0 \)

\( k \) = "slack" value ..., \( x_i \) must be outside this allowance to grow \( C^+ \) or \( C^- \)

\[ \frac{H}{2} \] typically to detect a \( 5\sigma \) shift

\[ \mu_0 + H = \mu_0 + h\sigma = UCL \]

\[ \mu_0 - H = \mu_0 - h\sigma = LCL \]

* Typical value \( H = 5\sigma \)

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Sample Cusum Status Chart.
Cumulative Chart Design:

- \( K = R_5 \) \( \ldots \) pick based on ARL performance (many studies exist, see Montgomery)
- \( H = h_5 \)

Typically \( K = \frac{1}{2} \) (close to minimizes ARL_1)

- \( h = 4 \)
  - ARL_1 = 8.38 samples to detect 10-shift
  - \( k_2 = \frac{1}{2} \)
  - \( h = 5 \)
  - ARL_1 = 10.4

<table>
<thead>
<tr>
<th>Shift in Mean (multiple of ( \sigma ))</th>
<th>( h = 4 )</th>
<th>( h = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>168</td>
<td>468</td>
</tr>
<tr>
<td>0.25</td>
<td>74.2</td>
<td>139</td>
</tr>
<tr>
<td>0.50</td>
<td>70.6</td>
<td>58.0</td>
</tr>
<tr>
<td>0.75</td>
<td>53.3</td>
<td>39.4</td>
</tr>
<tr>
<td>1.00</td>
<td>38.3</td>
<td>10.4</td>
</tr>
<tr>
<td>1.50</td>
<td>4.75</td>
<td>9.25</td>
</tr>
<tr>
<td>2.00</td>
<td>1.34</td>
<td>2.04</td>
</tr>
<tr>
<td>2.50</td>
<td>2.62</td>
<td>3.11</td>
</tr>
<tr>
<td>3.00</td>
<td>2.19</td>
<td>2.87</td>
</tr>
<tr>
<td>4.00</td>
<td>1.71</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Note: \( ARL_0 = 168 \) for \( h = 4 \)

False alarm performance:

\[
\frac{1}{1-\beta} = ARL_0
\]

Values of \( k \) and the Corresponding Values of \( h \) That Give \( ARL_0 = 370 \) for the Two-Sided Tabular Cusum (from Hawkins (1993a))

<table>
<thead>
<tr>
<th>( k )</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
<th>1.25</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
<td>8.01</td>
<td>4.77</td>
<td>3.34</td>
<td>2.52</td>
<td>1.99</td>
<td>1.61</td>
</tr>
</tbody>
</table>
Multi-Point Charts and Detection of Shifts

• The principle underlying the CUSUM and other charts that look at multiple historical points is the Detection of a Shift

• When we have multiple points, the principles of Maximum Likelihood Estimation (MLE) underlie our statistical inferences.

\[ \text{MLE} \]

The correct choice of the pdf moments maximizes the collective likelihood of the observations.

If \( x \sim f(x; \theta) \) with unknown \( \theta \), then \( \theta \) estimated by solving

\[
\max_{\theta} \left[ \prod_{i=1}^{m} \text{pdf} (x_i; \theta) \right]
\]

⇒ good for both estimation and detection (comparisons).

Example: Estimate mean of a normal distribution given observations \( x_i \), \( i = 1, 2, \ldots, m \)

\[
\max_{\mu_{\text{est}}} \left[ \prod_{i=1}^{m} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x_i - \mu_{\text{est}}}{\sigma} \right)^2} \right]
\]
**MLE Control Charts**

The goal is to detect if the process state has changed from some "good" state to some bad point:

\[ f_G \quad \sim \quad f_B \]

\[ x_i \]

Calculate \( \sum_{i=1}^{m} \log \frac{f_B(x_i)}{f_G(x_i)} \Rightarrow \) large when process in the \( f_B \) distribution/state

Use "summing" statistic \( S_m \)

\[ S_m = \sum_{i=1}^{m} \log \frac{f_B(x_i)}{f_G(x_i)} > L \text{ as signal} \]

That is, when "enough" evidence mounts that we are in the bad state (determined by \( L \) threshold), signal alarm.

**Cusum:** When \( G \) is mean of a Normal distribution, then \( S_m \) simplifies to

\[ S_m = \sum_{i=1}^{m} (\bar{x}_i - \mu_0) > L, \text{ } L \text{ depends on shift to be detected!} \]

**Key Point:** Use of Cusum and related charts requires knowledge about \( f_B \).

- MLE approaches are very general, and can be applied to deviations in means and variances, and also to multivariate cases (means and covariances).
Acceptance Charts

- Issue: What if we have an extremely capable process, Cpk >> 1? We may only want to be concerned when the process gets close to our spec limits.

![Distribution of process output](image1)

(A)

![Distribution of the sample mean \( \bar{X} \)](image2)

(B)

Control limits on a modified control chart. (A) Distribution of process output. (B) Distribution of the sample mean \( \bar{X} \).

- Approach: Build chart equivalent to hypothesis test:

\[ H_0: \mu_L \leq \mu \leq \mu_U \]

where we set \( \mu_L \) and \( \mu_U \) so that some \( \delta \) fraction nonconforming is okay:

\[ \mu_L = LSL + Z_\delta \delta \]

where \( Z_\delta \) is upper 100(1- \delta) percentage pt.

\[ \mu_U = UCL - Z_\delta \delta \]
• Chart design: For a type I error:

\[
UCL = \mu_U + \frac{Z_{\alpha/2} \sigma}{\sqrt{n}}
\]

\[
= USL - \frac{Z_{\alpha/2} \sigma}{\sqrt{n}} - USL \left( Z_{\alpha} - \frac{Z_{\alpha/2}}{\sqrt{n}} \right) \sigma
\]

\[
LCL = \mu_L - \frac{Z_{\alpha/2} \sigma}{\sqrt{n}}
\]

\[
= LSL + \frac{Z_{\alpha/2} \sigma}{\sqrt{n}} - LSL \left( Z_{\alpha} - \frac{Z_{\alpha/2}}{\sqrt{n}} \right) \sigma
\]

* Can use 3 for \( \alpha/2 \) if desired

• Caveat: depends on well-controlled \( \sigma \)

\Rightarrow should also monitor using R or S chart!