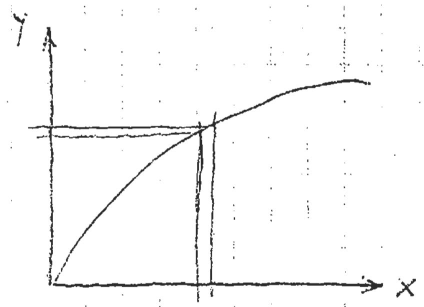
6.866
10/14/04

Sensitivity to error



x = unknown parameter (world)

y = measurement (image)

$$y = f(x) \qquad \delta y = \frac{df}{dx}\,\delta x$$

ex: $y = \sin\alpha x$

$f' = \alpha\cos\alpha x$

$$\delta x = \frac{1}{\alpha\cos\alpha x}\,\delta y = \frac{1}{\alpha\sqrt{1-y^2}}\,\delta y$$

$\longrightarrow$ problem if $y \sim 1$

$$x = f^{-1}(y)$$

$$\delta x = \frac{df^{-1}}{dy}\,\delta y$$

$$f^{-1}(y) = \frac{1}{\alpha}\sin^{-1}(y)$$

$$\frac{df^{-1}}{dy} = \frac{1}{\alpha}\,\frac{1}{\sqrt{1-y^2}}$$

Variance:

$$\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{N}\delta x_i^2$$

$$\sigma_y^2 = \frac{1}{N}\sum_{i=1}^{N}\delta y_i^2$$

$$\sigma_x^2 = \frac{1}{(df/dx)^2}\,\sigma_y^2$$

$$\boxed{\sigma_x = \frac{1}{\| df/dx \|}\,\sigma_y}$$

Noise in weighted sum

$$x = \sum_{i=1}^{M} w_i x_i \qquad \left(\sum_{1}^{N} w_i = 1 \quad \text{in general} \right)$$

Noise $\quad x + e = \sum_{i=1}^{M} w_i (x_i + e_i)$

$$e = \sum_{i=1}^{M} w_i e_i$$

expected value of x : $\quad \langle x \rangle = \sum_{i=1}^{M} w_i \langle x_i \rangle \qquad$ mean: μ

$$e^2 = \sum_{i=1}^{M} w_i e_i \sum_{j=1}^{M} w_j e_j$$

$$e^2 = \sum_{i=1}^{M} w_i w_j e_i e_j$$

$$e_i e_j = 0 \quad \text{if } i \neq j$$
$$= e_i^2 \quad \text{if } i = j \quad \Rightarrow \quad e^2 = \sum_{i=1}^{n} w_i^2 e_i^2$$

$$\sigma_x^2 = \sum_{i=1}^{n} w_i^2 \sigma_{x_i}^2$$

e.g.: $w_i = \frac{1}{N}$

$$\sigma_x^2 = \frac{1}{N^2} \sum_{i=1}^{n} \sigma_{x_i}^2 = \frac{1}{N} \sigma_{x_i}^2 \quad \Rightarrow \quad \boxed{\sigma_x = \frac{1}{\sqrt{N}} \sigma_{x_i}}$$

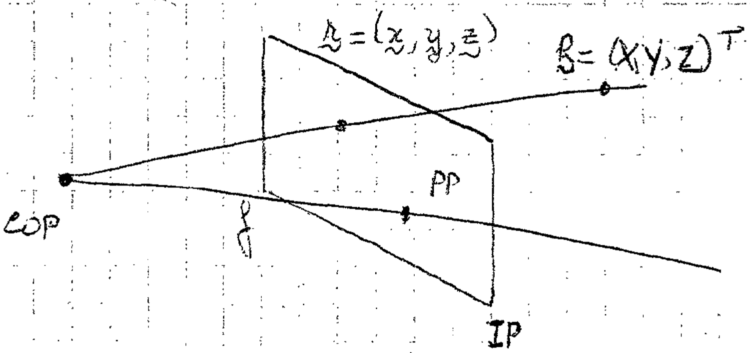
Notation

vectors = bold face $\underset{\sim}{a} = (x, y, z)^T$

unit vectors: $\hat{\underset{\sim}{x}}$ $\hat{\underset{\sim}{y}}$ $\hat{\underset{\sim}{z}}$

dot product: $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a}^T \cdot \underset{\sim}{b}$

dyadic product: $\underset{\sim}{a} \cdot \underset{\sim}{b}^T = \begin{vmatrix} a_x b_x & a_x b_y & a_x b_z \\ & \dots & \end{vmatrix}$



$$\begin{cases} \frac{x}{f} = \frac{X}{Z} \\ \frac{y}{f} = \frac{Y}{Z} \end{cases} \qquad \frac{z}{f} = 1$$

$$\boxed{\frac{1}{f}\,r = \frac{1}{R\cdot\hat{z}}\,R}$$

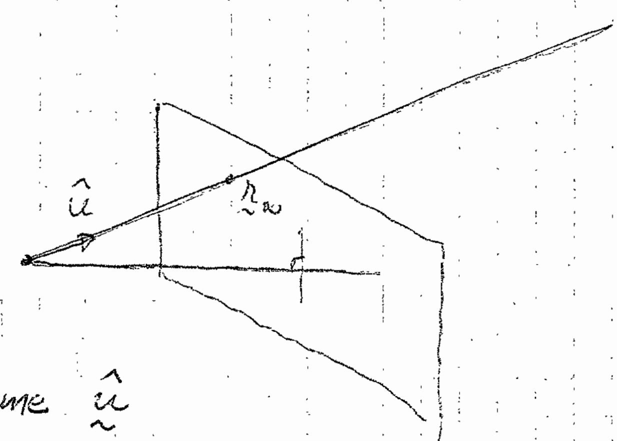
In 3D, $R = R_0 + s\hat{u}$

ref. point — unit vector direction

$$\frac{1}{f}\,r = \frac{1}{(R_0 + s\vec{u})\cdot\hat{z}}\cdot(R_0 + s\hat{u})$$

For any s, we can get r.

Vanishing point, $s \to \infty$



$$\boxed{\frac{1}{f}\,r_\infty = \frac{1}{\vec{u}\cdot\hat{z}}\,\hat{u}}$$

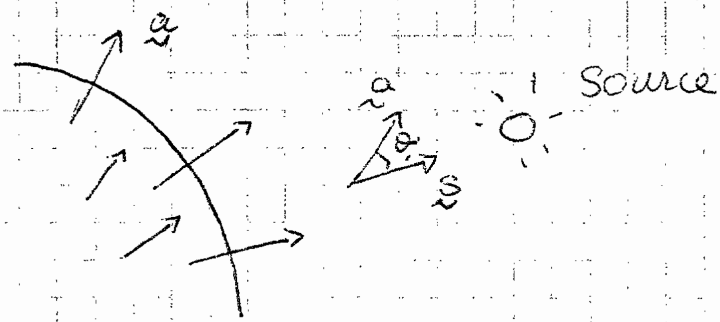
All parallel lines have the same $\hat{u}$
thus the same vanishing point.

Point on the image: $\dfrac{\hat{u}}{\hat{u}\cdot\hat{z}} + \dfrac{1}{R_0\cdot\hat{z}}$

Example: $\underset{\sim}{a} \cdot \underset{\sim}{s} = \alpha$ ← image brightness

$\underset{\sim}{b} \cdot \underset{\sim}{s} = \beta$

$\underset{\sim}{c} \cdot \underset{\sim}{s} = \gamma$



$$\begin{vmatrix} \underset{\sim}{a}^T \\ \underset{\sim}{b}^T \\ \underset{\sim}{c}^T \end{vmatrix} \cdot \underset{\sim}{s} = \begin{vmatrix} \alpha \\ \beta \\ \gamma \end{vmatrix} \qquad \underset{\sim}{s} = M^{-1} \cdot \begin{vmatrix} \alpha \\ \beta \\ \gamma \end{vmatrix} \qquad \det(M) \neq 0$$

$$M^{-1} = \frac{1}{[\underline{a}\ \underline{b}\ \underline{c}]} \begin{vmatrix} \underline{b} \times \underline{c} & \underline{c} \times \underline{a} & \underline{a} \times \underline{b} \end{vmatrix}$$

triple product: $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c} = \det(M) = \text{volume}$

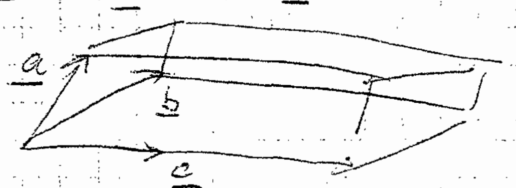
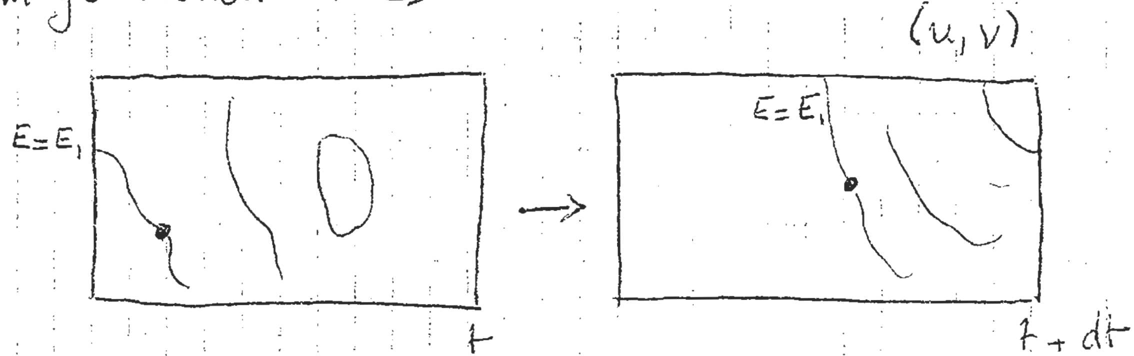


Image motion in 2D



Track isophotes ($E = E_1$)

$E(x, y, t)$

$E(x, y, t) = E$ constant brightness assumption

$E(x + \delta x, y + \delta y, t + \delta t) = E(x, y, t)$

$\frac{\partial E}{\partial x} \delta x + \frac{\partial E}{\partial y} \delta y + \frac{\partial E}{\partial t} \delta t + \varepsilon(\|x, y, t\|)$

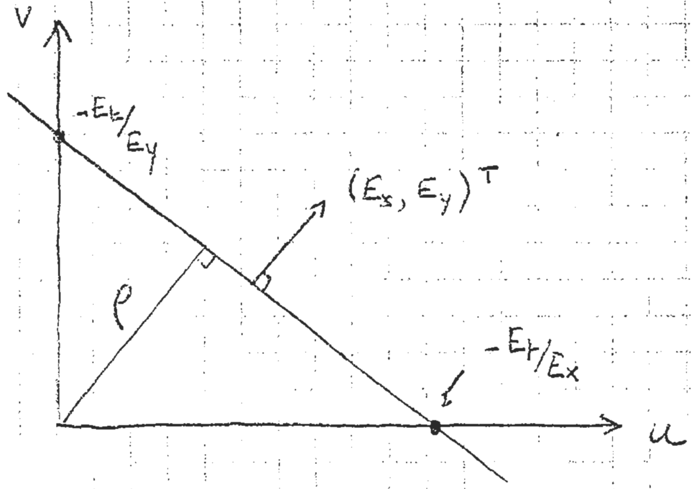
$\left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y} \right)$ = brightness gradient

$u = \frac{dx}{dt}$

$v = \frac{dy}{dt}$

$$\boxed{u E_x + v E_y + E_t = 0}$$

constraint equation



$$(u, v) \circ (E_x, E_y)^T = -E_t$$

$$(u, v) \circ \frac{1}{\|(E_x, E_y)\|} (E_x, E_y)^T = \underbrace{\frac{-1}{\|E_x, E_y\|} E_t}_{\rho}$$

In two points,

$$\begin{cases} u E_{x_1} + v E_{y_1} + E_{t_1} = 0 \\ u E_{x_2} + v E_{y_2} + E_{t_2} = 0 \end{cases}$$

$$\begin{vmatrix} E_{x_1} & E_{y_1} \\ E_{x_2} & E_{y_2} \end{vmatrix} \begin{vmatrix} u \\ v \end{vmatrix} = - \begin{vmatrix} E_{t_1} \\ E_{t_2} \end{vmatrix}$$

$$\begin{vmatrix} u \\ v \end{vmatrix} = \frac{1}{E_{x_1} E_{y_2} - E_{x_2} E_{y_1}} \begin{vmatrix} E_{y_2} & -E_{y_1} \\ -E_{x_2} & E_{x_1} \end{vmatrix} \begin{vmatrix} E_{t_1} \\ E_{t_2} \end{vmatrix}$$