Simple Types

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Before we Start
Some more Coq
Induction over natural numbers

\[ N ::= O \mid S\ N \]

**Induction principle:**
To prove \( \forall n \in N.\ P(n) \):

- **Base case:**
  
  *Show* \( P(0) \).

- **Inductive case:**
  
  *Assume* \( P(n) \).
  
  *Show* \( P(S(n)) \).
Structural Induction

\[ T ::= \text{Leaf} \mid \text{Node } T \; T \]

**Induction principle:**
To prove \( \forall \; t \in T. \; P(t) \):

**Base case:**
*Show* \( P(\text{Leaf}) \).

**Inductive case:**
*Assume* \( P(t1) \).
*Assume* \( P(t2) \).
*Show* \( P(\text{Node } t1 \; t2) \).
Another Example

\[ E ::= \text{Const N} \mid \text{Plus } E \ E \mid \text{Times } E \ E \]

**Induction principle:**
To prove \( \forall e \in E. \ P(e) \):

**Base case:**
Show \( P(\text{Const n}) \).

**Inductive case 1:**
Assume \( P(e_1) \).
Assume \( P(e_2) \).
Show \( P(\text{Plus } e_1 \ e_2) \).

**Inductive case 2:**
Assume \( P(e_1) \).
Assume \( P(e_2) \).
Show \( P(\text{Times } e_1 \ e_2) \).
Proofs as a Datatype

\[
\begin{array}{c|c|c}
\text{even}(0) & \text{even}(n) & \text{even}(n+2) \\
\text{even}(0) & \text{even}(0) & \text{even}(2) \\
\text{even}(0) & \text{even}(2) & \text{even}(4) \\
\end{array}
\]

Example Derivations:

\[
\text{even} ::= \text{Even0} : \text{even}(0) \\
| \text{Even2} (\text{even } n) : \text{even}(n+2)
\]

Examples:

- Even0 : even(0)
- Even2(Even0) : even(2)
- Even2(Even2(Even0)) : even(4)

...and so on for all even numbers.
**Induction on Proofs (Rule Induction)**

\[
\begin{array}{c|c}
\text{even(0)} & \text{even(n)} \\
\hline
\text{even(n+2)} & \text{even(n+2)} \\
\end{array}
\]

**even ::= Even0 : even(0) \mid Even2 (even n) : even(n+2)**

**Induction principle:**
To prove \( \forall n \in \mathbb{N}. \text{even}(n) \Rightarrow P(n) \):

**Base case:**
Show \( P(0) \).

**Inductive case:**
Assume \( P(n) \).
Show \( P(n+2) \).

Because I have a rule that if \( n \) is even, it lets me prove that \( n+2 \) is even.

Also called Induction on the Structure of Derivations
More Rule Induction

\[
\begin{array}{c|c|c}
\text{eval} & \text{eval}(e_1, n_1) & \text{eval}(e_2, n_2) \\
\hline
\text{eval} (\text{Const } n, n) & \text{eval}(\text{Plus } e_1 e_2, n_1 + n_2) \\
\end{array}
\]

**eval** ::= **EvConst**: \text{eval} (\text{Const } n, n) \\
| **EvPlus**: (\text{eval}(e_1, n_1)) (\text{eval}(e_2, n_2)) \\
| : \text{eval} (\text{Plus } e_1 e_2, n_1 + n_2)

**Induction principle:**
To prove \( \forall e \in E, n \in N. \text{eval } e \ n \Rightarrow P(e, n) \):

**Base case:**

*Show* \( P(\text{Const } n, n) \).

**Inductive case:**

*Assume* \( P(e_1, n_1) \).

*Assume* \( P(e_2, n_2) \).

*Show* \( P(\text{Plus } e_1 e_2, n_1 + n_2) \).
More Tactics

- **induction N:**
  - Induction on the derivation of the [N]th hypothesis in the conclusion
  - (numbering goes left to right and starts at 1).
- **destruct E**
  - Do case analysis on the constructor used to build term [E].
- **assumption**
  - Prove a conclusion that matches a known hypothesis; like doing apply H where H is the known hypothesis.
- **eapply thm**
  - Like apply, but leaves placeholders for theorem parameters that are not known yet.
- **eassumption**
  - Like assumption, but also learns values for placeholders in the process.
- **rewrite <- H**
  - Like [rewrite], but rewrites right-to-left.
More powerful tactics

• generalize thm1,...,thmN
  - Bring the statements of a set of theorems into the goal explicitly so that other tactics don't need to deduce them manually.

• firstorder
  - Magic heuristic procedure for proofs based on first-order logic rules.
  - (It's undecidable in general, so don't get too excited.)
And now some types!
Why Types

```haskell
let
  f x = if x then 5 else 2
in
  f 5+1

let
  f x = if x then 5 else 2
in
  f 6

let
  f x = if x then 5 else 2
in
  if 6 then 5 else 2
```

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What to do in this situation?

**Options**

1) Leave it up to the implementation
   - that’s the C approach
   - is it a good idea?

2) Provide a mechanism to identify and rule out such “bad” programs
   - programs can only run if you can prove they will execute to completion according to the semantics of the language
   - type systems will allow us to do this!

3) Prescribe correct behavior for every program
   - untyped λ-calculus works like this
   - do any practical languages do this?
   - type systems are useful in this situation too.
Self-application and Paradoxes

Self application, i.e., \( (x \ x) \) is dangerous.

Suppose:
\[ u \equiv \lambda y. \text{if } (y \ y) = a \text{ then } b \text{ else } a \]
What is \( (u \ u) \)?
\[ (u \ u) \rightarrow \text{if } (u \ u) = a \text{ then } b \text{ else } a \]

Contradiction!!!

This was one of the original motivations for types
What is a type system

• Narrow View
  – It’s a mechanism for ensuring that variables only take values from predefined sets
    • Ex. Integers, Strings, Characters
  – A mechanism for avoiding unchecked errors
    • by ruling out programs with undefined behaviors
    • by specifying how a program should fail (e.g. NullPointerException)

• Expansive View
  – It’s a light-weight proof system and annotation mechanism for efficiently checking for a specific property of interest
  – Address bugs that go beyond corner-cases in the semantics
    • Information flow violations
    • deadlocks
    • etc, etc, etc
What are Types?

- A method of classifying objects (values) in a language

\[ x :: \tau \]

says object \( x \) has type \( \tau \) or object \( x \) belongs to a type \( \tau \)

- \( \tau \) denotes a set of values.

This notion of types is different from types in languages like C, where a type is a storage class specifier.
Type Correctness

- If $x :: \tau$ then only those operations that are *appropriate* to set $\tau$ may be performed on $x$.

- A program is *type correct* if it never performs a wrong operation on an object.
  
  - Add an *Int* and a *Bool*
  - Head of an *Int*
  - Square root of a *list*
Type Safety

• A language is *type safe* if only *type correct* programs can be written in that language.

• Most languages are *not* type safe, i.e., have “holes” in their type systems.

*Fortran*: Equivalence, Parameter passing  
*Pascal*: Variant records, files  
*C, C++*: Pointers, type casting

*However, Java, Ada, CLU, ML, Id, Haskell, Bluespec, etc. are type safe.*
Type Declaration vs Reconstruction

- Languages where the user must declare the types
  - CLU, Pascal, Ada, C, C++, Fortran, Java

- Languages where type declarations are not needed and the types are reconstructed at run time
  - Scheme, Lisp

- Languages where type declarations are generally not needed but allowed, and types are reconstructed at compile time
  - ML, Id, Haskell, pH, Bluespec

A language is said to be **statically typed** if type-checking is done at compile time
Polymorphism

• In a monomorphic language like Pascal, one defines a different length function for each type of list.

• In a polymorphic language like ML, one defines a polymorphic type (list t), where t is a type variable, and a single function for computing the length.

• Haskell and most modern functional languages have polymorphic types and follow the Hindley-Milner type system.

Simple types = Non polymorphic types

more on polymorphic types – next time …
Formalizing a Type System
Formalizing a type system

- The type system is almost never orthogonal to the semantics of the language
  - The types in a program can affect its behavior (e.g. operator overloading)

- We don’t define the type system in isolation, we define a typed *language* including definitions of
  - The syntax
  - dynamic semantics (e.g. operational semantics)
  - static semantics
    - also known as typing rules
    - describe how types are assigned to elements in a program
  - type soundness argument
    - describe the relationship between static and dynamic semantics
Basic notation

• The type system assigns types to elements in the language
  – basic notation: \( e : T \) (\( e \) is of type \( T \))
  – What is the type of :
    \[ 5 \]
  – An environment associates types with free variables
  – This is called a Judgment

• The types of some elements depends on the environment
  – basic notation \( \Gamma \vdash e : T \)
    (Given environment \( \Gamma \), we can derive that \( e \) is of type \( T \))
  – Ex.
    \[ x : \text{int}, y : \text{int} \vdash x + y : \text{int} \]
Static Semantics

• Typing rules
  – Typing rules tell us how to derive typing judgments
  – Very similar to derivation rules in Big Step OS

\[
\frac{\text{premises}}{\text{Judgment}}
\]

• Ex. Language of Expressions

\[
\frac{x: T \in \Gamma}{\Gamma \vdash x : T} \quad \frac{\Gamma \vdash N : \text{int}}{\Gamma \vdash e1 + e2 : \text{int}}
\]

\[
\frac{\Gamma \vdash e1 : \text{int} \quad \Gamma \vdash e2 : \text{int}}{\Gamma \vdash e1 + e2 : \text{int}}
\]
Ex. Language of Expressions

\[
\begin{align*}
\frac{x: T \in \Gamma}{\Gamma \vdash x : T} & \quad \frac{\Gamma \vdash N : int}{\Gamma \vdash e_1 : int} & \frac{\Gamma \vdash e_2 : int}{\Gamma \vdash e_1 + e_2 : int}
\end{align*}
\]

- Show that the following Judgment is valid

\[
x: \text{int}, y: \text{int} \vdash x + (y + 5) : \text{int}
\]

\[
\frac{x: \text{int}, y: \text{int} \vdash x : \text{int}}{}
\frac{x: \text{int}, y: \text{int} \vdash (y + 5) : \text{int}}{}
\frac{x: \text{int}, y: \text{int} \vdash x + (y + 5) : \text{int}}{}
\]

\[
\frac{x: \text{int} \in x: \text{int}, y: \text{int}}{}
\frac{x: \text{int}, y: \text{int} \vdash y : \text{int}}{}
\frac{x: \text{int}, y: \text{int} \vdash 5 : \text{int}}{}
\frac{x: \text{int}, y: \text{int} \vdash (y + 5) : \text{int}}{}
\frac{x: \text{int}, y: \text{int} \vdash x + (y + 5) : \text{int}}{}
\]
Simply Typed $\lambda$ Calculus ($F_1$)

- Basic Typing Rules

\[
\begin{align*}
\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} & \quad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e) : \tau_1 \to \tau_2} & \quad \frac{\Gamma \vdash e_1 : \tau' \to \tau \quad \Gamma \vdash e_2 : \tau'}{\Gamma \vdash e_1 e_2 : \tau}
\end{align*}
\]

- Extensions

\[
\begin{align*}
\frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}} & \quad \frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 = e_2 : \text{bool}}
\end{align*}
\]

\[
\begin{align*}
\frac{\Gamma \vdash e : \text{bool} \quad \Gamma \vdash e_t : \tau \quad \Gamma \vdash e_f : \tau}{\Gamma \vdash \text{if } e \text{ then } e_t \text{ else } e_f : \tau}
\end{align*}
\]
Example

- Is this a valid typing judgment?

\[ \vdash (\lambda x: \text{bool} \ \lambda y: \text{int} \ \text{if } x \text{ then } y \text{ else } y + 1): \text{bool} \to \text{int} \to \text{int} \]

- How about this one?

\[ \vdash (\lambda x: \text{int} \ \lambda y: \text{bool} \ x + y): \text{int} \to \text{bool} \to \text{int} \]
Example

- What’s the type of this function?

\[(\lambda f. \lambda x. \text{if } x = 1 \text{ then } x \text{ else } (f \ f \ (x-1)) \ast x)\]

\[
\begin{align*}
\frac{x: \tau \in \Gamma}{\Gamma \vdash x : \tau} & \quad \quad \\
\frac{\Gamma, x: \tau_1 \vdash e: \tau_2}{\Gamma \vdash (\lambda x: \tau_1 \ e): \tau_1 \to \tau_2} & \quad \quad \\
\frac{\Gamma \vdash e_1: \tau' \to \tau \quad \Gamma \vdash e_2: \tau'}{\Gamma \vdash e_1 e_2: \tau} & \quad \quad \\
\frac{\Gamma \vdash e_1: \text{int} \quad \Gamma \vdash e_2: \text{int}}{\Gamma \vdash e_1 + e_2: \text{int}} & \quad \quad \\
\frac{\Gamma \vdash e: \text{bool} \quad \Gamma \vdash e_t: \tau \quad \Gamma \vdash e_f: \tau}{\Gamma \vdash \text{if } e \text{ then } e_t \text{ else } e_f: \tau} & \quad \quad \\
\frac{\Gamma \vdash e_1: \text{int} \quad \Gamma \vdash e_2: \text{int}}{\Gamma \vdash e_1 = e_2: \text{bool}} & \quad \quad \\
\end{align*}
\]

- Hint: This IS a trick question
Simply Typed $\lambda$ Calculus ($F_1$)

- We have defined a really strong type system on $\lambda$-calculus
  - It’s so strong, it won’t even let us write non-terminating computation
  - We can actually prove this!