Term Models & Equational Completeness

We can generalize the theory of Arithmetic Expressions developed in Notes 2 to other algebraic structures.

1 First-order Terms

A signature, Σ, specifies the names of operations and the number of arguments (arity) of each operation. For example, the signature of arithmetic expressions is the set of names {0, 1, +, *, −} with + and * each of arity two, and − of arity one. The constants 0 and 1 by convention are considered to be operations of arity zero.

As a running example, we will consider a signature, Σ₀, with three names: F of arity two, G of arity one, and a constant, c.

The First-order Terms over Σ are defined in essentially the same way as arithmetic expressions. We’ll omit the adjective “first-order” in the rest of these notes.

Definition 1.1. The set, \( T_\Sigma \), of Terms over Σ are defined inductively as follows:

- Any variable, \( x \), is in \( T_\Sigma \).\(^1\)
- Any constant, \( c \in \Sigma \), is in \( T_\Sigma \).
- If \( f \in \Sigma \) has arity \( n > 0 \), and \( M_1, \ldots, M_n \in T_\Sigma \), then

\[
\left(f \left( M_1, \ldots, M_n \right) \right) \in T_\Sigma.
\]

For example,

\[
c, \ x, \ F(c, x), \ G(G(F(y, G(c)))
\]

are examples of terms over \( \Sigma_0 \).

\(^1\)We needn’t specify exactly what variables are. All that matters is that variables are distinct from other kinds of terms and from operation names in the signature.
2 Substitution

A substitution over signature, \( \Sigma \), is a mapping, \( \sigma \), from the set of variables to \( T_\Sigma \). The notation

\[
[x_1, \ldots, x_n := M_1, \ldots, M_n]
\]

describes the substitution that maps variables \( x_1, \ldots, x_n \) respectively to \( M_1, \ldots, M_n \), and maps all other variables to themselves.

**Definition 2.1.** Every substitution, \( \sigma \), defines a mapping, \( [\sigma] \), from \( T_\Sigma \) to \( T_\Sigma \) defined inductively as follows:

\[
\begin{align*}
x[\sigma] & := \sigma(x) & \text{for each variable, } x. \\
c[\sigma] & := c & \text{for each constant, } c. \\
f(M_1, \ldots, M_n)[\sigma] & := f(M_1[\sigma], \ldots, M_n[\sigma]) & \text{for each } f \in \Sigma \text{ of arity } n > 0.
\end{align*}
\]

For example

\[
F(G(x), y)[x, y := F(c, y), G(x)] \text{ is the term } F(G(F(c, y)), G(x)).
\]

3 Models

A model assigns meaning to terms by specifying the space of values that terms can have and the meaning of the operations named in a signature. Models are also called “first-order structures” or “algebras.”

**Definition 3.1.** A model, \( \mathcal{M} \), for signature, \( \Sigma \), consists a nonempty set, \( A_\mathcal{M} \), called the carrier of \( \mathcal{M} \), and a mapping \( [\cdot]_0 \) that assigns an \( n \)-ary operation on the carrier to each symbol of arity \( n \) in \( \Sigma \). That is, for each \( f \in \Sigma \) of arity \( n > 0 \),

\[
[f]_0 : A_\mathcal{M}^n \to A_\mathcal{M},
\]

and for each \( c \in \Sigma \) of arity \( 0 \),

\[
[c]_0 \in A_\mathcal{M}.
\]

For example, a model for \( \Sigma_0 \) might have carrier equal to the set of binary strings, with \( F \) meaning the concatenation operation, \( G \) meaning reversal, and \( c \) meaning the symbol 1.

**Definition 3.2.** An \( \mathcal{M} \)-valuation, \( V \), is a mapping from variables into the carrier, \( A_\mathcal{M} \).

Once we have a model and valuation, we can define a value from the carrier for any term, \( M \). The meaning, \( [M]_{\mathcal{M}} \), of the term itself is defined to be the function from valuations to the term’s value under a valuation. We’ll usually omit the subscript when it’s clear which model, \( \mathcal{M} \), is being referenced.
**Definition 3.3.** The meaning, $[M]_M$, of a term, $M$, in a model, $M$, is defined by structural induction on the definition of $M$:

$[x]V := V(x)$ for each variable, $x$.
$[c]V := [c]_0$ for each constant, $c \in \Sigma$.
$[f(M_1, \ldots, M_n)]V := [f]_0([M_1]V, \ldots, [M_n]V)$ for each $f \in \Sigma$ of arity $n > 0$.

**Definition 3.4.** For any function, $F$, and elements $a, b$, we define the patch of $F$ at $a$ with $b$, in symbols, $F[a \leftarrow b]$, to be the function, $G$, such that

$$G(x) = \begin{cases} b & \text{if } x = a. \\ F(x) & \text{otherwise.} \end{cases}$$

The fundamental relationship between substitution and meaning is given by

**Lemma 3.5 (Substitution).**

$[M[x := N]]V = [M](V[x \leftarrow [N]V))$,

Lemma 3.5. follows by structural induction on $M$ as in Notes 3.

**Definition 3.6.** An equation is an expression of the form $(M = N)$ where $M, N$ are terms. The equation is valid in $M$, written

$M \models (M = N)$,

iff $[M] = [N]$. If $E$ is a set of equations, then $E$ is valid in $M$, written

$M \models E$,

iff $M \models (M = N)$ for each equation $(M = N) \in E$.

Finally, $E$ logically implies another set, $E'$, of equations, written

$E \models E'$,

when $E'$ is valid for any model in which $E$ is valid. That is, for every model, $M$

$M \models E$ implies $M \models E'$.

**4 Proving Equations**

There are some standard rules for proving equations over a given signature from any set, $E$, of equations. The equations in $E$ are called the axioms. We write $E \vdash E$ to indicate that equation $E$ is provable from the axioms. The proof rules are given in Table 1.

Note that a more general (congruence) rule follows from the rules above. Namely, let $\sigma_1$ and $\sigma_2$ be substitutions and define

$E \vdash (\sigma_1 = \sigma_2)$

to mean that $E \vdash (\sigma_1(x) = \sigma_2(x))$ for all variables, $x$. 

Table 1: Standard Equational Inference Rules.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Rightarrow E$ for $E \in \mathcal{E}$.</td>
<td>(axiom)</td>
</tr>
<tr>
<td>$\Rightarrow (M = M)$.</td>
<td>(reflexivity)</td>
</tr>
<tr>
<td>$(M = N) \Rightarrow (N = M)$.</td>
<td>(symmetry)</td>
</tr>
<tr>
<td>$(L = M), (M = N) \Rightarrow (L = N)$.</td>
<td>(transitivity)</td>
</tr>
<tr>
<td>$(M_1 = N_1), \ldots, (M_n = N_n) \Rightarrow (f(M_1, \ldots, M_n) = f(N_1, \ldots, N_n))$</td>
<td>(congruence)</td>
</tr>
<tr>
<td>for each $f \in \Sigma$ of arity $n &gt; 0$.</td>
<td></td>
</tr>
<tr>
<td>$(M = N) \Rightarrow (M[x := L] = N[x := L])$.</td>
<td>(substitution)</td>
</tr>
</tbody>
</table>

**Lemma 4.1.** If $\mathcal{E} \vdash (\sigma_1 = \sigma_2)$, then

$\mathcal{E} \vdash (M[\sigma_1] = M[\sigma_2])$.  

(general congruence)

There is also a more general (substitution) rule:

**Lemma 4.2.** If $\mathcal{E} \vdash (M = N)$, then for any substitution, $\sigma$,

$\mathcal{E} \vdash (M[\sigma] = N[\sigma])$.  

(general substitution)

**Problem 1.** Prove that the (general congruence) rule implies (congruence).

**Problem 2.** (a) Prove the (general congruence) rule of Lemma 4.1.
(b) Prove the (general substitution) rule of Lemma 4.2. *Hint:* Prove that any substitution into a term $M$ can be obtained by a series of one-variable substitutions, namely,

$M[\sigma] = M[x_1 := N_1][x_2 := N_2] \ldots [x_n := N_n]$

for some variables $x_1, x_2, \ldots, x_n$ and terms $N_1, N_2, \ldots, N_n$. There is slightly more to the proof than might be expected.

**Theorem 4.3 (Soundness).** If $\mathcal{E} \vdash (M = N)$, then $\mathcal{E} \models (M = N)$.

*Proof.* The Theorem follows by induction on the structure of the formal proof that $(M = N)$. The only nontrivial case is when $(M = N)$ is a consequence of the (substitution) rule. We show that this case follows from the Substitution Lemma 3.5.

Namely, suppose $M$ is $(M'[x := L])$ and $N$ is $(N'[x := L])$ where $\mathcal{E} \vdash (M' = N')$. Then by induction,

$\mathcal{E} \models (M' = N')$.  

(1)
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So if \( M \) is any model such that \( M \models \mathcal{E} \), we have by definition from (1) that
\[
[M'] = [N'].
\] (2)

Now,
\[
[M]V = [M'[x := L]]V \quad \text{(by def of } M')
\]
\[
= [M'](V[x \leftarrow [L]V)) \quad \text{(by Subst. Lemma 3.5)}
\]
\[
= [N'](V[x \leftarrow [L]V)) \quad \text{(by (2))}
\]
\[
= [N'[x := L]]V \quad \text{(by Subst. Lemma 3.5)}
\]
\[
= [N]V \quad \text{(by def of } N')
\]

which shows that \([M] = [N]\), and hence \( \mathcal{E} \models (M = N) \), as required.

\[\square\]

5 Completeness

We are now ready to prove

**Theorem 5.1 (Completeness).** If \( \mathcal{E} \models (M = N) \), then \( \mathcal{E} \vdash (M = N) \).

We prove Theorem 5.1 by constructing a model, \( \mathcal{M}_\mathcal{E} \), in which provable equality and semantical equality coincide. Namely, we will show that

**Lemma 5.2.**

\( \mathcal{E} \vdash (M = N) \iff \mathcal{M}_\mathcal{E} \models (M = N) \).

Completeness follows directly from Lemma 5.2. In particular, since \( \mathcal{E} \vdash E \) by the (axiom) rule for any equation, \( E \in \mathcal{E} \), Lemma 5.2 immediately implies that

\[ \mathcal{M}_\mathcal{E} \models \mathcal{E} \].

Moreover, if \( \mathcal{E} \not\vdash (M = N) \), then \( \mathcal{M}_\mathcal{E} \not\models (M = N) \), and so \( \mathcal{E} \not\models (M = N) \).

It remains to define the model, \( \mathcal{M}_\mathcal{E} \), and to prove Lemma 5.2. The model \( \mathcal{M}_\mathcal{E} \) will be a term model depending only on the axioms \( \mathcal{E} \), not on the particular terms \( M \) or \( N \).

The proof rules of (reflexivity), (symmetry) and (transitivity) imply that for any fixed \( \mathcal{E} \), provable equality between terms \( M \) and \( N \) is an equivalence relation. We let \( [M]_\mathcal{E} \) be the equivalence class of \( M \) under provable equality, that is,

\[ [M]_\mathcal{E} := \{ N \mid \mathcal{E} \vdash (M = N) \} \].

So we have by definition
\[
\mathcal{E} \vdash (M = N) \iff [M]_\mathcal{E} = [N]_\mathcal{E}.
\] (3)
The carrier of $\mathcal{M}_\mathcal{E}$ will be defined to be the set of $[M]_\mathcal{E}$ for $M \in \mathcal{T}_\Sigma$. The meaning of constants, $c \in \Sigma$, will be

$$[c]_0 := [c]_\mathcal{E}.$$ 

The meaning of operations $f \in \Sigma$ of arity $n > 0$ will be

$$[f]_0([M_1]_\mathcal{E}, \ldots, [M_n]_\mathcal{E}) := [f(M_1, \ldots, M_n)]_\mathcal{E}.$$ 

Notice that $[f]_0$ applied to the equivalence classes $[M_1]_\mathcal{E}, \ldots, [M_n]_\mathcal{E}$ is defined in terms of the designated terms $M_1, \ldots, M_n$ in these classes. To be sure that $[f]_0$ is well-defined, we must check that the value of $[f]_0$ would not change if we chose other designated terms in these classes. That is, we must verify that

$$\text{if } [M_1]_\mathcal{E} = [N_1]_\mathcal{E}, \ldots, [M_n]_\mathcal{E} = [N_n]_\mathcal{E}, \text{ then } [f(M_1, \ldots, M_n)]_\mathcal{E} = [f(N_1, \ldots, N_n)]_\mathcal{E}.$$ 

But this is an immediate consequence of the (congruence) rule. 

For any substitution, $\sigma$, let $V_\sigma$ be the $\mathcal{M}$-valuation given by

$$V_\sigma(x) := [\sigma(x)]_\mathcal{E}$$

for all variables, $x$. The following key property of the term model follows by structural induction on terms, $M$.

**Lemma 5.3.**

$$[M]V_\sigma = [M[\sigma]]_\mathcal{E}.$$ 

**Problem 3.** Prove Lemma 5.3.

Now let $\iota$ be the identity substitution $\iota(x) := x$ for all variables, $x$. For any term, $M$, the substitution instance $M[\iota]$ is simply identical to $M$, so Lemma 5.3 immediately implies

$$[M]V_\iota = [M]_\mathcal{E}.$$ 

In particular, if $[M] = [N]$, then $[M]_\mathcal{E} = [N]_\mathcal{E}$, so from equation (3), we conclude Lemma 5.2 in the right-to-left direction:

**Corollary 5.4.** If $[M]_{\mathcal{M}_\mathcal{E}} = [N]_{\mathcal{M}_\mathcal{E}}$, then $\mathcal{E} \vdash (M = N)$.

Finally, for the left-to-right direction of Lemma 5.2, we prove

**Corollary 5.5.** If $\mathcal{E} \vdash (M = N)$, then $[M]_{\mathcal{M}_\mathcal{E}} = [N]_{\mathcal{M}_\mathcal{E}}$.

**Proof.** If $\mathcal{E} \vdash (M = N)$, then by (general substitution) $\mathcal{E} \vdash (M[\sigma] = N[\sigma])$ for any substitution, $\sigma$. So

$$[M[\sigma]]_\mathcal{E} = [N[\sigma]]_\mathcal{E}$$

(4)
by (3). Now let \( V \) be any \( \mathcal{M} \)-valuation, and let \( \sigma \) be any substitution such that \( \sigma(x) \in V(x) \) for all variables, \( x \). This ensures that

\[
V = V_\sigma, \quad (5)
\]

and we have

\[
\begin{align*}
[M]V &= [M[\sigma]]_\varepsilon \quad \text{(by (5) & Lemma 5.3)} \\
 &= [N[\sigma]]_\varepsilon \quad \text{(by (4))} \\
 &= [N]V \quad \text{(by (5) & Lemma 5.3)}.
\end{align*}
\]

Since \( V \) was an arbitrary valuation, we conclude that \([M] = [N]\), as required. \( \square \)