1 Minimum Cost Circulation Problem

Theorem 1 Let $f$ be a circulation. The following are equivalent:

(i) $f$ is of minimum cost.

(ii) $G_f$ has no negative cost directed cycles.

(iii) $\exists p: c_p(v, w) \geq 0 \ \forall (v, w) \in E_f$, where $c_p(v, w) = c(v, w) + p(v) - p(w)$.

Proof: $i \Rightarrow ii$ and $iii \Rightarrow i$ were proven last lecture. All that remains is the proof of $ii \Rightarrow iii$:

Let $G'$ be obtained from the residual graph $G_f$ by adding a vertex $s$ linked to all other vertices by edges of cost 0 (the costs of these edges do not matter). Let $p(v)$ be the length of the shortest path from $s$ to $v$ in $G'$ with respect to the costs. These quantities are well-defined since $G_f$ does not contain any negative cost directed cycles, and every vertex is reachable from $s$. By definition of the shortest paths, $p(w) = p(u) + c(u, w) \ \forall (v, w) \in E_f$.

This implies that $c_p(v, w) > 0 \ \forall (v, w) \in E_f$. \hfill $\Box$

2 Klein’s Algorithm for MCCP

Klein’s Cycle canceling algorithm:

1. Let $f$ be any circulation.

2. While $G_f$ contains a negative cycle $\Gamma$ do

   push $\delta = \min_{(v, w) \in \Gamma} u_f(v, w)$ along $\Gamma$.

Argument for Correctness:

If the algorithm terminates, then the circulation found must be optimum. Furthermore, if all capacities and costs are integers, then the algorithm will terminate.

Why?

- $f(v, w)$ is always an integer, thus $\delta = \min_{(v, w) \in \Gamma} u_f(v, w) \geq 1$

- If $|c(v, w)| \leq C$ and $|f(v, w)| \leq U$, then the absolute value of the cost of the optimal circulation is at most $mCU$

Therefore, the algorithm terminates after $O(mCU)$ iterations.

Remark 1 If the edge capacities in the graph are irrational, then the algorithm is not correct.

The cycle canceling algorithm can be applied to the Max-Flow Problem by making appropriate modifications to the graph $G$. Let $G'$ be obtained by setting the cost of all edges within $G$ to 0. Furthermore, select two vertices $s$ and $t$ from within the graph, and add the directed edges $(s, t)$ and $(t, s)$, where $c(s, t) = 1, c(t, s) = -1$ and both edges have infinite capacity. Now, solving for
the maximum flow between $s$ to $t$ is equivalent to solving for the minimum cost circulation, which contains $s$ and $t$. In this circumstance, Klein’s Algorithm reduces to the Ford-Fulkerson Algorithm.

**Ford-Fulkerson Augmenting Path Algorithm:**

1. Begin with zero flow: $f = 0$.
2. While $G_f$ contains a directed path $P$ from $s$ to $t$ do
   push $\delta = \min_{(v, w) \in P} u_f(v, w)$ along $P$.

The running time given above for Klein’s Cycle-Canceling Algorithm is not polynomial. The negative cost cycle in Klein’s Algorithm (or the directed path in the Ford-Fulkerson Algorithm) must be chosen appropriately to insure a polynomial running time.

**Candidates for Cycles in Klein’s Algorithm:**

1. The most negative cost cycle in $G_f$?
   *Finding this cycle is an NP-Hard problem, so it would not be a viable choice.*

2. The negative cycle in $G_f$ which would yield the maximum cost improvement?
   *Finding this cycle is again an NP-Hard problem.*
   However for the Max-Flow Problem, this choice reduces to finding the st-path with maximum residual capacity. Such a path can be found in $O(m)$ time, $m = |E|$. The resulting Max-Flow algorithm is known as the “fattest” path algorithm (Edmonds-Karp ’72). The number of iterations necessary is $O(m \log U)$, thus the running of the algorithm is $O(m^2 \log U)$.

3. Minimum Mean-Cost Cycle?
   Define the mean cost of a cycle $\Gamma$ to be:
   \[
   \mu(f) = \min_{\text{cycles} \Gamma \in G_f} \frac{c(\Gamma)}{|\Gamma|}
   \] (1)
   where $|\Gamma|$ denotes the number of edges in $\Gamma$. The minimum mean cost of all cycles of the residual graph $G_f$ would thus be:
   \[
   \mu(f) = \min_{\text{cycles} \Gamma \in G_f} \frac{c(\Gamma)}{|\Gamma|}
   \] (2)
   The minimum mean-cost cycle can be determined in strongly polynomial time by using a modified version of the Bellman-Ford Algorithm. More precisely, the minimum mean cost cycle can be found in $O(mn)$ time. Using this method to solve the Min-Cost Circulation Problem yields the Goldberg-Tarjan Algorithm, which runs in polynomial time. Using this method to solve the Max-Flow Problem yields what is known as the “shortest” augmenting path algorithm (Edmonds-Karp). This Max-Flow Algorithm is able to find the augmenting path in $O(m)$ time, and requires $O(mn)$ iterations to arrive at the solution. Thus, the total running time is $O(m^2 n)$.  

10-2
3 The Goldberg-Tarjan Algorithm

Goldberg-Tarjan Algorithm:

1. Begin with zero flow: \( f = 0 \).
2. While \( \mu(f) < 0 \) do
   push \( \delta = \min_{(v,w) \in \Gamma} u_f(v, w) \) along a minimum mean cost cycle \( \Gamma \) of \( G_f \).

Analysis of Goldberg-Tarjan Algorithm:

In order to analyze this algorithm, it is necessary to define the concept of proximity measure for a circulation \( f \).

Definition 1 A circulation \( f \) is \( \epsilon \)-optimal if there exists \( p \) such that \( c_p(v, w) \geq -\epsilon \) \( \forall (v, w) \in E_f \).

Definition 2 \( \epsilon(f) = \text{minimum } \epsilon \text{ such that } f \text{ is } \epsilon \)-optimal.

The following theorem states that the minimum mean cost \( \mu(f) \) of all cycles in \( G_f \) is equal to \( -\epsilon(f) \), as defined above.

Theorem 2 For any circulation \( f \), \( \mu(f) = -\epsilon(f) \).

Proof:

- \( \mu(f) \geq -\epsilon(f) \)
  By definition, there exists \( p \) such that \( c_p(v, w) \geq -\epsilon(f) \) \( \forall (v, w) \in E_f \). Thus, it is implied that \( c_p(\Gamma) \geq -\epsilon(f) |\Gamma| \) for any directed cycle \( \Gamma \in G_f \). But for any \( \Gamma \in G_f \), \( c(\Gamma) = c_p(\Gamma) \). Thus, dividing both sides by \( |\Gamma| \) yields that the mean cost of any directed cycle \( \Gamma \in G_f \) is at least \( -\epsilon(f) \). Therefore, \( \mu(f) \geq -\epsilon(f) \).

- \( \epsilon(f) \leq -\mu(f) \)
  Consider \( \mu(f) \). For every cycle \( \Gamma \in G_f \), it is the case that \( \frac{c(\Gamma)}{|\Gamma|} \geq \mu(f) \). Let \( c'(v, w) = c(v, w) - \mu(f) \) \( \forall (v, w) \in E_f \). With respect to this new cost function \( c' \) every cycle \( \Gamma \in G_f \) will have nonnegative cost. Now, let \( G' \) be obtained by adding a new node \( s \) to \( G_f \) and adding directed edges from \( s \) to \( v \) \( \forall v \in V \), all with zero cost. Let \( p(v) \) be the cost with respect to \( c' \) of the shortest path from \( s \) to \( v \) in the new graph \( G' \). For all edges \( (v, w) \), \( p(w) \leq p(v) + c'(v, w) = p(v) + c(v, w) - \mu(f) \). This implies that \( c_p(v, w) \geq \mu(f) \) \( \forall (v, w) \in E_f \).
  Therefore, \( \epsilon(f) \leq -\mu(f) \).

- \( \mu(f) \geq -\epsilon(f) \) and \( \epsilon(f) \leq -\mu(f) \) \( \Rightarrow \epsilon(f) = -\mu(f) \).

Remark 2 Along the minimum mean cost cycle \( \Gamma \), \( c_p(v, w) = -\epsilon(f) \) \( \forall (v, w) \in \Gamma \).

Having completed the necessary definitions and proofs, we may now proceed with the analysis of the Goldberg-Tarjan Algorithm. The following theorem considers only one iteration of the algorithm.

Theorem 3 Let \( f \) be a circulation and let \( f' \) be the circulation obtained by canceling the minimum mean cost cycle \( \Gamma \) of \( G_f \). Then, \( \epsilon(f') \leq \epsilon(f) \).

Proof: By definition, there exists \( p \) such that \( c_p(v, w) \geq -\epsilon(f) \) \( \forall (v, w) \in E_f \). In the case of the minimum mean cost cycle \( \Gamma \) of \( G_f \), \( c_p(v, w) = -\epsilon(f) \) \( \forall (v, w) \in \Gamma \). After performing the one cycle-canceling step, we obtain the new residual graph \( G_{f'} \). We claim that \( c_p(v, w) \geq -\epsilon(f) \) \( \forall (v, w) \in E_{f'} \). In the case of all edges \( (v, w) \in E_f \cap E_f \), the claim is certainly true. In the case of all edges
$(v, w) \in E_f \setminus E_f$, it must be true that $(w, v) \in \Gamma$. For all $(w, v) \in \Gamma$, $c_p(w, v) = -\epsilon(f)$, and thus $c_p(v, w) = \epsilon(f) \geq -\epsilon(f)$. Therefore, $c_p(v, w) \geq -\epsilon(f)$ holds true for all $(v, w) \in E_f$. \hfill \Box

The above theorem shows that by completing a single iteration of the Goldberg-Tarjan algorithm, it is impossible to generate a new flow which is farther from optimality than the original.