In the last lecture the concept of the dynamic trees data structure was introduced and a number of operations that dynamic trees must support were described. In this lecture, we define the data structure in detail and describe the efficient implementation of operations using expose, an extended splay operation.

1 Dynamic Trees

Dynamic trees are a data structure intended for maintaining a representation of a set of rooted trees, and performing a number of operations (discussed below) on these trees. In today's lecture we will be using the example tree depicted in figure 1 to describe dynamic trees and the expose operation which is central to the efficient implementation of operations on dynamic trees.

We view rooted trees as unions of node-disjoint paths. This divides the edges of the tree into two sets. Solid edges are those that are on the node-disjoint paths that the tree is composed of, and dashed edges are those that are not on these paths. Figure 2 shows a possible partitioning of the example tree into a set of node disjoint paths. Note that each path consisting of solid edges is a directed path from top to bottom. We refer to the top of each such path as the tail, and the bottom as the head (for example h in the path from h to e is the tail and e the head).

2 Virtual Trees

The union of disjoint paths described above can be used to represent virtual trees. In a virtual tree, each solid path is represented by a binary search tree such that the following two conditions hold:

1 A successor node in the binary search tree is a parent in the rooted tree.
2 The root of a binary search tree is linked to the parent of the tail of the path it corresponds to in the rooted tree.

For example, figure 3 shows the virtual tree corresponding to the union of disjoint paths shown in figure 2.

There are three kinds of edges in a virtual tree, corresponding to the three types of children a node can have. Left and right children of a node are connected to the node by solid edges, and middle children of a node are connected to it by dashed edges. Thus a node can either have no parent, a parent connected to it by a solid line, or a parent connected to it by a dashed line. Note that there can be many virtual trees corresponding to a rooted tree, because there are two different degrees of freedom involved in constructing a virtual tree — the union of disjoint paths could be different, as could the structure of the binary search trees corresponding to the paths.

3 The expose Operation

The expose(v) operation is an extended splay operation on virtual trees. The important parts of this operation are to make sure that the path from v to the root is solid and that the binary search tree representing the path to which v belongs is rooted at v. We will describe the process in 3 steps although it can be implemented somewhat more efficiently in a single step (although the asymptotic efficiency is the same).
Figure 1: Example tree

Figure 2: Tree as a union of disjoint paths

Figure 3: A virtual tree corresponding to the example
3.1 Step 1

Step 1 consists of walking from \( v \) to the root of the virtual tree. Whenever the walk enters a binary search tree (solid edges) at some node \( w \), a splay(\( w \)) operation is performed, bringing \( w \) to the root of that tree. Middle children are not affected in this step. For example in the expose(\( j \)) operation on the example tree, we perform 3 splay operations, on \( j \), \( f \) and \( c \) respectively, leading to the tree shown in figure 4. Note that at the end of step 1 of an expose(\( v \)) operation, \( v \) will be connected to the root of the virtual tree only by dashed edges.

3.2 Step 2: Splicing

Step 2 consists of walking from \( v \) to the root of the virtual tree while exchanging the subtree rooted at \( v \) with the left subtree of the parent of \( v \) at each step, as illustrated in figure 5 (why the solid tree remains a binary search tree is left as an exercise). The new tree of the example for expose(\( j \)) after step 2 is performed can be seen in figure 6. Note that at the end of this step, there will be a solid path from the root of the tree to the node being exposed.

3.3 Step 3

Step 3 consists of walking from \( v \) to the root in the virtual tree, splaying \( v \) to the root. The example tree after the completion of step 3 (and the entire expose operation is shown in figure 7. Note that in the analysis, we can charge the entire cost of step 2 to the final splaying operation in step 3.
Figure 6: After step 2 (splicing) of expose(j)

Figure 7: After step 3 of expose(j)
4 Operations on Dynamic Trees

4.1 Maintaining Cost Information

Some of the operations we need to perform on dynamic trees involve costs of edges and paths. At each node $v$, we could maintain the minimum cost in the subtree rooted at $v$, ignoring the virtual tree and dashed edges, but this is problematic for the $\text{add} - \text{cost}$ operation. So instead we maintain two things at each node $x$, namely

$$\Delta \min(x) = \text{cost}(x) - \mincost(x)$$

where $\mincost(x)$ represents the smallest cost along the path corresponding to the BST in the original tree, and:

$$\Delta \cost(x) = \begin{cases} 
\text{cost}(x) & \text{if } x \text{ is the root of a BST,} \\
\text{cost}(x) - \text{cost}(p(x)) & \text{otherwise (}p(x)\text{ denotes the parent of } x)\end{cases}$$

Therefore, if $x$ is the root of a BST, then $\text{cost}(x) = \Delta \cost(x)$ and $\mincost(x) = \Delta \cost(x) - \Delta \min(x)$.

$\Delta \min(x)$ and $\Delta \cost(x)$ can both be updated in $O(1)$ time when one does a rotation or a splice. It is important that both $v$ and $w$ are roots of BSTs in the splicing step.

4.2 Implementation of Operations

$\text{find} - \text{cost}(v)$: First, $\text{expose}(v)$. Now $v$ is the root, so return $\Delta \cost(v)$.

$\text{find} - \text{root}(v)$: $\text{expose}(v)$. Then follow right children until you reach a leaf $w$ of the BST. $\text{splay}(w)$, then return $w$.

$\text{find} - \text{min}(v)$: $\text{expose}(v)$. Examining $\Delta \min$ and $\Delta \cost$ go down to the minimum, say $w$, ignoring the left subtree of $v$. $\text{splay}(w)$

$\text{add} - \text{cost}(v, x)$: $\text{expose}(v)$. Add $x$ to $\Delta \cost(v)$ and subtract $x$ from $\Delta \cost(\text{left}(v))$.

$\text{cut}(v)$: $\text{expose}(v)$. Add $\Delta \cost(v)$ to $\Delta \cost(\text{right}(v))$. Remove the edge $(v, \text{right}(v))$.

$\text{link}(v, w, x)$: $\text{expose}(v)$, then $\text{expose}(w)$. Set $w$ to be a middle child of $v$. 

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