Recall: packing of n squares without rotation into a square is strongly NP-complete \( [L2] \)

**Edge-unfolding polyhedra:** given a polyhedron, cut along edges to unfold flat without overlap

- not always possible \( [\text{Biedl et al. & Bern et al. 1998}] \)
- strongly NP-hard \( [\text{Abel & Demaine 2011}] \)
even for orthogonal polyhedra topologically sphere

**Reduction from Square Packing:**

- infrastructure: polyhedron with square with tower with squares & "atoms" on side
- "pipe" is super long but can move out
  \( \Rightarrow \) squares must pack inside base of tower
- atoms are universal: can turn left/right/straight in 2D unfolding & left/right/straight on tower surface
  \( \Rightarrow \) can connect & place squares as in any (slightly perturbed) packing, then exit via pipe
- lots of details e.g. shrink squares slightly to enable perturbation
Snake cube puzzle: AKA Cubra circa 1990
- given chain of unit cubes each with specified “turn angle” of 0 or 90° (elastic through centers)
- goal: fold it into larger cube (exactly)
- NP-hard

[Abel, Demaine, Demaine, Eisenstat, Lynch, Scharddl 2012]

Reduction from 3-Partition:
- infrastructure:
  - fill cube to leave \( x \times y \times z \) box
  - fill box to leave “hub & slots” shape
  - each hub is \( 8 \times t \times huge \)
- \( a_i \) gadget: \( 8a_i \) must go in 1 hub
  - 8 to avoid coming back to same \( 4 \times 4 \times 4 \) voxel
- connected together by zig-zag gadget
- zig-zag is universal:
  - \( 2 \times 2 \times 2 \) can turn/go straight
    \( \Rightarrow \) fill Hamiltonian shapes scaled \( 2 \times 2 \times 2 \)
  - \( 2 \times 2 \times 2 \) refinement makes any shape Hamiltonian
    \( \Rightarrow \) \( 4 \times 4 \times 4 \) refinement makes fillable by zig-zag
- parity issue: snake alternates in cell parity
- claim: can start & end at any faces of opposite parity
Disk packing: pack n given disks into given shape

- motivation: computational origami design
  (tree method — see Lang)
- strongly NP-hard [Demaine, Fekete, Lang—OSME 2010]

Reduction from 3-Partition:
- infrastructure:
  - build \( n/3 \) symmetric \( 8 \) pockets
  - equilateral \( \Delta \): forced packing
  - square target: forced packing
    + repeated subdivision with forced packings
    + fill all other pockets by repeatedly adding maximal disks, until small enough (depth \( \approx \lg n \))
- triple gadget: (in symmetric pocket)
  - scale \( a_i \)'s & \( t \) so that \( t = 1 \)
  - shrink center disk by \( -1/N \)
  - shrink \( a_i \) disk by \( -1/N^2 \), \( \Rightarrow \) big
  - grow it by \( +a_i/N \)
- key property: disks fit \( \iff \) \( a_i + a_j + a_k \leq t \)
  (proof by geometry + Taylor series)
Clickomania: [Schuessler ~2000?]
- given rectangular grid of colored squares
- move = remove connected group of >1 square of the same color
- remaining squares fall within each column
- empty columns disappear

- polynomial for one row or column
- reduces to CFG parsing
- NP-hard for
  - 2 columns & 5 colors
  - 5 columns & 3 colors
- OPEN: 2 rows? 2 colors?

Reduction from 3-Partition:
- left column mostly checkerboard except middle & interspersed red $\square$s to measure t's
- collapses $\iff$ red $\square$s removed
- right column has $a_i$ groups + red squares on top
- details: spacing out groups & reds while still getting alignment
Tetris: [Alexey Pazhitnov 1985]
- rectangular board
- tetromino blocks come one at a time
  - 4 unit squares joined edge-to-edge
- can rotate block as it falls from sky
- filled lines disappear
- stack to sky \( \Rightarrow \) die

- perfect information version:
  - know entire sequence of pieces to come
  - initial board position given
- NP-complete to [Breukelaar, Demaine, Hohenberger, Hoogeboom, Kosters, Liben-Nowell 2003]
  - survive
  - approximate \# lines/Tetrises/time until death
    up to a factor of \( n^{1-\varepsilon} \)

Reduction from 3-Partition \( \Rightarrow \) necessary: encoding in unary
- initial board = \( \frac{n}{3} \) buckets of “depth” \( \varepsilon \)
- \( a_i \) encoded as \( \begin{tabular}{c}
  \( \text{\( \boxed{\text{I}} \)} \)
  \( \text{\( \boxed{\text{I}} \)} \)
  \( \text{\( \boxed{\text{I}} \)} \)
\end{tabular} \)\( a_i \)
  \( \text{\( \boxed{\text{I}} \)} \)
  \( \text{\( \boxed{\text{I}} \)} \)
- claim: entire gadget must go in one bucket
- finale = \( \begin{tabular}{c}
  \( \text{\( \boxed{\text{I}} \)} \)
  \( \text{\( \boxed{\text{I}} \)} \)
  \( \text{\( \boxed{\text{I}} \)} \)
\end{tabular} \)\( n^{\varepsilon/4}+4 \)
1-planarity: draw a given graph in the plane such that each edge crosses at most 1 other

- NP-complete

Reduction from 3-Partition:
- uncrossable edge gadget:
- double wheel gadget:
  - unique embedding
  - one for A
  - one for triples
  - separate triples with thick edges every t hours around triples gadget
- $a_i$-gadget:
  A center triples center
GeoLoop & Ivan's Hinge puzzles: piano-hinged dissection

⇒ NP-complete from 3-Partition

Ruler folding:
- given carpenter's ruler with lengths $a_1, a_2, \ldots, a_n$
- goal: fold to fit in 1D box of length $L$

- weakly NP-complete [Hopcroft, Joseph, Whitesides-1985]
- pseudopolynomial (like 2-Partition)

Reduction from (2-)Partition:
- idea: Partition solvable ⇔ can assign signs to $a_i$'s such that $\sum_i \pm a_i = 0$
- folding flips sign; unfolding leaves sign
⇒ can fold ends together ⇔ Partition solvable
- construction: $2B, B, a_1, a_2, \ldots, a_n, B, 2B$

⇒ $2B$'s will be aligned & fit inside length-$2B$ box
⇒ can fold ends together ⇔ Partition solvable
Map folding (simple): given crease pattern, can it fold flat by sequence of simple folds?

- weakly NP-hard [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena - 2000]

for orthogonal paper & orthogonal creases
or square paper & 45° orthogonal creases

Reduction from Partition:
- similar to Ruler Folding
- 2 vertical creases check y extent against frame
- horizontal creases done before or after check
  if ruler folded
  5 if not

- force square paper into orthogonal shape:

OPEN: strongly NP-hard? pseudopolynomial?