3SUM: [Gajentaan & Overmars - CGTA 1995]
given n integers, do any 3 sum to 0?
(allowing same integer to be chosen > once)
- conjecture: no $O(n^{2-\varepsilon})$ algorithm "truly subquadratic"
- $O(n^2)$ randomized algorithm:
  - compute all pairwise sums
  - look in hash table of all negations
- $O(n^2)$ deterministic algorithm:
  - presort integers
  - for each target sum (negated integer):
    - advance left pointer right if too small
    - advance right pointer left if too big
- $O(n + u \log u)$ via FFT if integers $\in [-u,u]$
- $O(n^2/(\log n)^3)$ randomized in word RAM
  [Baran, Demaine, Pătraşcu-Alg.2008]
- $O(n^2/(\log n)^3)$ det.: $O(n^2/(\log n)^{2/3})$ rand. in real RAM
  $O(n^{1.5/\sqrt{\log n}})$ in decision tree model
  [Grønlund & Pettie - FOCS 2014]
**k-SUM**: given \( n \) integers, do any \( k \) sum to \( 0 \)?
- \( O(n^{\frac{k}{k-1}}) \) randomized algorithm
- conjecture: no \( O(n^{\frac{k}{k-1}-\epsilon}) \) algorithm
- \( \text{NP-complete for } k \) an input (\( \approx \text{Partition} \))
- \( \text{W[1]-hard w.r.t. } k \) (but quadratic parameter blowup from Clique \( \Rightarrow n^{o(k^2)} \) lower bound)
- ETH \( \Rightarrow \) no \( n^{o(k)} \) algorithm for \( k\text{-SUM} \) \( \leq n^{0.99} \) [Pătraşcu & Williams- SODA 2010]

**3SUM-hard** = \( O(n^{2-\epsilon}) \) algorithm \( \Rightarrow \) one for \( 3\text{SUM} \)

**3SUM reduction** = \( O(1) \)-call reduction on \( n'=O(n) \) running in \( O(n^{2-\epsilon}) \) time
- A \( 3\text{SUM}-\text{hard} \) (e.g. \( 3\text{SUM} \)) \( \Rightarrow \) \( \text{B 3SUM-hard} \)

**Base 3SUM-hard problems**: (all equivalent)
- \( 3\text{SUM} \) with \( u=n^3 \) via hashing [Pătraşcu- STOC 2010]
  \( \approx \) [Baran, Demaine, Pătraşcu-Alg. 2008]
- **Distinct 3SUM**: \( \exists 3 \) distinct integers summing to \( 0 \)?
  - reduction from \( 3\text{SUM} \): also check for doubled/tripled ints
  - reverse reduction?? [Mikhail Ruday, today]
- \( 3\text{SUM'} \): given sets \( A, B, C \) of \( n \) integers
  - \( \exists a \in A, b \in B, c \in C \) such that \( a+b=c \)?
  - reduction from \( 3\text{SUM} \): \( A=B=S, C=-S \) \( \text{(or } a+b+c=0 \text{)} \)
  - also reduction in reverse direction [Gajentaan & Overmars- CGTA 1995]
- **GeomBase**: given \(n\) points in 2D with \(y \in \{0,1,2\}\)
  - nonhorizontal line hitting 3 points?
  - reduction from/to 3SUM:
    - \(a \in A \iff (a,0)\)
    - \(b \in B \iff (b,2)\)
    - \(c \in C \iff (c/2,1)\)
    - \(a+b=c\)
    - \(c/2 = a+b/2\)

  [Gajentaan & Overmars - CGTA 1995]

**More 3SUM-hard problems:**

- also solvable in \(O(n^3)\) time

**3 points on a line**: given \(n\) points in the plane, are any 3 collinear?
  - reduction from **Distinct 3SUM**
  - \(x \in S \rightarrow (x, x^3)\)!

**Point on 3 lines**: given \(n\) lines in the plane, do any 3 meet at a point?
  - projective plane dual of 3 points on line:
    - \((a, b) \iff ax + by + 1 = 0\)
    - (lines \(ax+by=0\) passing through origin map to points @ infinity ~ avoid these)
    - preserves point/line incidence

**d-D versions**: \((d+1)\)-SUM hard
Separator: given n segments, is there a line splitting them into 2 nonempty groups?
- reduction from GeomBase
- if allow half-infinite segments, can all be horizontal (Sep. 1)
- else horizontal & vertical segments (Sep2)

Strips cover box: does union of n strips cover a given axis-aligned rectangle?
- reduction from GeomBase
- start from Separator 1 reduction
- rotated 90º
- dualize: \((m, b) \rightarrow y = mx + b\)
  - vertical segment \(\rightarrow\) strip
  - half-infinite segment \(\rightarrow\) half plane
- rectangle = bounding box of hexagonal hole in union of 6 half-planes
- restrict half planes to this rectangle
  \(\rightarrow\) 6 more strips
- uncovered point in dual
  = line in primal not hitting any segments
Triangles cover triangle:
- reduction from previous problem
- convert box $\rightarrow$ triangle with $O(1)$ strips
- split strips into 2 large $\Delta$s
- can assume $n$ triangles $\leq$ big triangle:
  - replace each triangle with intersection
  - triangulate resulting $O(1)$-gons

Hole in union: does union of $n$ triangles have a hole?
- reduction from previous problem ($\leq$ version)
- add thin $\Delta$s covering edges of big $\Delta$
- hole $\iff$ not covered
- reduction in reverse direction also possible

Triangle measure: area of union of $n$ triangles
- reduction from Triangles cover triangle ($\leq$)
- $\text{area(union)} = \text{area(big } \Delta) \iff \text{covered}$

Point covering: is there a $k$-way intersection between $n$ given half planes?
- reduction from Strips cover box
- strip $\rightarrow$ complement as 2 half planes
- rectangle $\rightarrow$ 4 half planes whose int. = rect.
- $k = n + 4$ (outside $n$ strips, inside rectangle)
Visibility between segments:
given n horizontal segments, is there a point on segment 1 that can see a point on segment 2 (unobstructed by segments)
- reduction from GeomBase like Separator 1

Visible triangle: given n horizontal triangles in 3D
can a given point see a point on triangle 1?
- reduction from Triangles cover triangle (view from infinity)
- reduction in reverse direction too

Planar motion planning: can you move the segment robot through horizontal & vertical segment obstacles?
- reduction from GeomBase (like Separator 1)

3D motion planning: can you translate vertical segment robot through horizontal Δ obstacles?
- reduction from Triangles cover triangle
- separate Δs slightly in z, in middle of cage
- goal: get from top half to bottom half of cage
- $O(n^2 \log n)$ algorithm
Fixed-angle chains: [Soss, Erickson, Overmars 2002]
which edge-spin operations cause collisions in a given fixed-angle chain?
- reduction from 3SUM'
- subtract 2M from each aeA \rightarrow A'
- add 2M to each ceC \rightarrow C'
\[ \max \text{ abs} \ (A' \cup B' \cup C') \]
- best algorithm: \( O(n^3) \) [Soss & Toussaint 2001]

Nonquadratic lower bounds: [Pătraşcu - STOC 2010]
- finding \( \Delta \) of prescribed weight in a weighted graph in \( O(E^{1.5-\varepsilon}) \) time is 3SUM-hard (as hard as \( O(n^{2-\varepsilon}) \) for 3SUM)
- finding \( |E| \Delta s \) in \( O(E^{4/3-\varepsilon}) \) time is 3SUM-hard
Conjectured cubic graph problems: (weighted)

Diameter: \( \max_{v,w} S(v,w) \) in undirected graph

- conjecture: no \( O(V^{3-\varepsilon}) \)-time algorithm
- no \( (3/2-\varepsilon) \)-approx. in \( O(E^{2-\varepsilon}) \) time,
even unweighted, assuming Strong ETH
  [Radity & Vassilevska Williams - T.ALG 2012]
- subcubic reduces to:
  \( \leftarrow O(n^{3-\varepsilon}) \)

APSP (All-Pairs Shortest Paths): \( S(v,w) \forall v,w \)
- \( O(V^3) \) via Floyd-Warshall algorithm
  (relax all edges \( |V| \) times)
- conjecture: no \( O(V^{3-\varepsilon}) \)-time algorithm
- APSP-hard = no \( O(V^{3-\varepsilon}) \) alg. assuming

Negative \( \Delta \): is there a 3-cycle of negative weight?
- APSP-hard \( \sim \) actually equivalent
- equivalent to listing \( |V| 1.99 \) negative \( \Delta \)s
- equivalent to testing \( \Delta \) inequality
  [Vassilevska Williams & Williams - FOCS 2010]

Radius: \( \min_{v} \max_{w} S(v,w) \)

Median: \( \min_{v} \sum_{w} S(v,w) \)

- APSP-hard \( \sim \) actually equivalent
  (directed or undirected)

[Abboud, Grandoni, Vassilevska Williams - SODA 2015]