PPAD: definition later - start with motivation

Motivation 1: Economic Game Theory

Game:
- n players 1, 2, ..., n
- for each player p: set $S_p$ of strategies
- payoff for each player p:
  $u_p : S_1 \times S_2 \times \cdots \times S_n \rightarrow \mathbb{R}$
- e.g. Penalty Shot Game

Nash Equilibrium = locally optimal product distribution of strategies for the players such that no one player can (by changing just their strategy) improve their expected payoff.

i.e. $x_1, x_2, \ldots, x_n$ such that $\forall p:\ E[u_p(x_1, \ldots, x_p, \ldots, x_n)] \geq E[u_p(x_1, \ldots, x'_p, \ldots, x_n)]
\forall x'_p \in D(S_p)$
- e.g. 1/2 - 1/2 strategies in Penalty Shot Game
- exist in 2-player zero-sum games [von Neumann 1928]
- via linear programming
- exist in n-player games [Nash 1950]
- still no poly-time algorithm to find them
Motivation 2: Brouwer's Fixed-Point Theorem
for any convex, closed, bounded set $S$,
any continuous map $f: S \rightarrow S$ has a
fixed point $p \in S : f(p) = p$ [Brouwer 1910]

Nash's proof via Brouwer's Theorem
- $f: [0,1]^n \rightarrow [0,1]^n$ is essentially a vector
  field indicating how each player can improve
  their mixed strategy (distribution)
- fixed point of $f =$ Nash equilibrium

Motivation 3: Sperner's Lemma
- square grid graph +
  backslash diagonals
- assign vertices 3 colors

2D version: if boundary is legally colored
then there are an odd number ($\geq 1$)
of trichromatic $\Delta$

d-dimensional version too (not covered here)
Proof of Brouwer via Sperner:
- for all $\varepsilon$, show approximate fixed point: $|f(x)-x| < \varepsilon$ via Sperner’s Lemma
- color points according to direction of $f(x)-x$ (which of 3 boundaries)
- use compactness to take limit $\varepsilon \to 0$ (may not preserve oddness of solution count)

**Computational version of Sperner:**
- grid of size $2^n \times 2^n$
- internal vertex colors given by circuit $C$
- boundary in canonical legal coloring
- goal: find trichromatic $\Delta$

**Computational version of Nash:**
- given # players $n$, enumeration of strategy set $S_p$ & utility function $u_p : S \to \mathbb{R}$ of every player $p$
- goal: $\varepsilon$-Nash equilibrium
  $\Rightarrow$ expected payoff can’t improve by more than $+\varepsilon$
- avoids representation issue for irrational equilibria (required for e.g. $n=3$ game)
Search problem defined by relation \( R \subseteq \{0, 1\}^* \times \{0, 1\}^* \)

where \((x, y) \in R\) means \(y\) is solution to \(x\)

Total if \(\forall x \exists y : (x, y) \in R\) i.e. always \(\exists \geq 1\) solution

- e.g. Sperner & Nash & Brouwer

\(\text{FNP} = \{\text{NP search problems}\}\)

\(\text{FNP-complete} \in \text{FNP} \& \exists\text{one-call (Karp) reduction}

from every problem \(\in\) \(\text{FNP}\)

- impossible for total problems

reducing from non-total problem e.g. SAT

Complexity theory for total problems: (TFNP)

- identify combinatorial argument for existence proof

- define complexity class

- check tightness via completeness result

Proof of Sperner’s Lemma:

- add artificial trichromatic \(\Delta\) at boundary

- define directed walk from that \(\Delta\):

  keep crossing bichromatic edges with same 2 colors
  with same orientation (else find trichromatic \(\Delta\))

- can’t exit square by valid boundary coloring

- can’t form a cycle \(\bigcirc\) (uncolorable)

- for odd number theorem: can walk from every
  other trichromatic \(\Delta\) to another \(\Rightarrow\) even #

except for one from boundary
Directed parity argument:
- vertices of graph represent Δs
- all vertices have in & out degrees ≤ 1
⇒ graph = disjoint union of directed paths, cycles, & isolated vertices
- degree-1 vertex = trichromatic Δ
- degree-2 vertex = walkable (2 bichromatic edges with right orientation)
- degree-0 vertex = rest

Nonconstructive step: if there's an unbalanced vertex then there's another in-deg. ≠ out-deg.

End of the Line:
- each vertex v has candidate incoming & outgoing edge P(v) & N(v)
  - given as circuit: V → (size 2^n)
- actual edge (v,w) ⇐ both ends agree:
  N(v) = w ∧ P(w) = v
- goal: if O^n is unbalanced, find another unbalanced node checkable in O(n) time (4 circuit evaluations)
- EFNP: certificate = another unbalanced node

PPAD = { search problems ∈ FNP reducible to [Papadimitriou 1994]
So: \[ \text{Nash} \rightarrow \text{Brouwer} \rightarrow \text{Sperner} \rightarrow \text{PPAD} \]

In fact: \[ \text{Nash} \leftarrow \text{Brouwer} \leftarrow \text{Sperner} \leftarrow \text{PPAD} \]

i.e.: \[ \text{Nash, Brouwer, Sperner are PPAD-complete} \]

\[ \text{[Papadimitriou 1994]} \]

\[ \text{[Daskalakis, Goldberg, Papadimitriou 2006]} \]

- even for 2-player Nash \[ \text{[Chen & Deng 2006]} \]

Proof sketch: generic PPAD

\[ \rightarrow \text{embed graph in } [0,1]^3 \]

\[ \rightarrow 3D \text{ Sperner} \]

\[ \rightarrow \text{Arithmetic Circuit SAT} \]

\[ \rightarrow \text{Nash} \]
Arithmetic Circuit SAT:

- **Input:** variable nodes $x_1, \ldots, x_n \leftarrow \text{in degree 1}

- **Gate nodes:** $\circlearrowright \rightarrow \circlearrowright \rightarrow$ etc. $\leftarrow \text{in degree } \in \{0, 1, 2\}$

- **Cycles allowed**

- **Arbitrary out degrees**

- **Goal:** assignment of values $\in [0, 1]$ to $x_1, \ldots, x_n$

- Satisfying all gate constraints:
  
  - $\circlearrowright \rightarrow \circlearrowright \rightarrow y \Rightarrow y = x$

  - $\circlearrowright \rightarrow \circlearrowright \rightarrow z = x + y$

  - $\circlearrowright \rightarrow \circlearrowright \rightarrow y \Rightarrow x = c$ \text{ for constant } c \in [0, 1]

  - $\circlearrowright \rightarrow \circlearrowright \rightarrow y \Rightarrow y = c \cdot x$

  - $\circlearrowright \rightarrow \circlearrowright \rightarrow z \Rightarrow z = \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x > y \\ \text{arbitrary if } x = y \end{cases}$

- **Total:** always a satisfying assignment

- **PPAD-complete**

- Improvement from exponential noise tolerance $\Rightarrow$ polynomial noise tolerance $\leftarrow$

  - Approximate Arith. Circuit SAT

- $\approx n^{-c}$
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