6.895 Theory of Parallel Systems  
Due: Thursday, November 27th  
Problems 12–15

**Problem 12.** Denote the butterfly network running from major cycles to minor cycles by $W$, let its “transpose” $W^T$ be the butterfly network running from minor cycles to major cycles. Thus, the Benes network is essentially the network $WW^T$. We’ve seen that $WW^T$ can route any permutation (offline) by node-disjoint paths.

**Problem 12.a.** Show that the network $WWW$, formed by 4 butterfly networks connected in tandem, can route any permutation through node-disjoint paths. (*Hint:* Show that $WW$ can implement the *bit-reverse* permutation.)

**Problem 12.b.** (*Hanel.* ) Show that the network $WWW$ can route any permutation by node-disjoint paths.

**Problem 12.c.** (*Open research problem.*) Can the network $WW$ route any permutation by node-disjoint paths?

**Problem 13.** Shuffling

**Problem 13.a.** Show that $r$ perfect out-shuffles return an $N = 2^d$-card deck to its original order if and only if $2^r \equiv 1 \pmod{N - 1}$.

**Problem 13.b.** Show that 8 perfect out-shuffles are necessary and sufficient to restore a 52-card deck to its original order.

**Problem 13.c.** How many perfect out-shuffles are necessary and sufficient to restore a 54-card deck to its original order?

**Problem 14.** De Bruijn

**Problem 14.a.** Construct a 32-bit de Bruijn sequence.

**Problem 14.b.** Prove that the bisection width of an $N$-node de Bruijn network is $O(N/\lg N)$. If you wish, you may use the fact that the bisection width of an $N$-node shuffle-exchange network is $O(N/\lg N)$. 
Problem 15. Kautz networks

A \((d, k)\) Kautz network is defined as follows:

- Each node is a word \(\langle x_1, x_2, \ldots, x_k \rangle\) on the alphabet \(\{0, 1, \ldots, d\}\), where \(x_i \neq x_{i+1}\) for \(i = 1, 2, \ldots, k - 1\).
- For each node \(\langle x_1, x_2, \ldots, x_k \rangle\) and all \(z \in \{0, 1, \ldots, d\} - \{x_k\}\), an edge exists from \(\langle x_1, x_2, \ldots, x_k \rangle\) to \(\langle x_2, \ldots, x_k, z \rangle\).

Problem 15.a. Draw a \((2, 3)\) Kautz network.

Problem 15.b. Prove that every node in a \((d, k)\) Kautz network has out-degree \(d\) and indegree \(d\). Prove (careful: two cases) that the diameter is \(k\).

Problem 15.c. How many nodes \(N\) does a \((d, k)\) Kautz network contain in terms of \(d\) and \(k\)? (Give a precise answer, not a bound.) Prove it.

Problem 15.d. Conjecture a good upper bound on the bisection width of a \((d, k)\) Kautz network. Optional: Prove your bound.

Problem 15.e. Some architects have recently become enamored with the idea of interconnecting the processors of a parallel computer with a \((d, k)\) Kautz network, typically with \(d \geq 4\). List salient pros and cons.