Systolic computation

E.g., linear array I/O-0-0-0-0-0...

"Fixed-connection" network
1. Underlying graph fixed
2. Local communication only
3. I/O location restricted

At each step of a globally synchronous clock,
each processor
1. receives inputs from neighbors (or I/O)
2. inspects local memory
3. performs local computation
4. updates local memory
5. generates outputs for neighbors

Example: Sorting

- Accept left input
- Compare input to stored value
- Store smaller value
- Output bigger # to right
Correctness: induction

$N$ inputs. How many steps? $2N = \Theta(N)$

"Discuss outputting of values."

Total time = $3N$

Sorting in the bit model (vs. word model)

- One processor per bit.
- $N$ k-bit #s

\[
\begin{array}{cccccc}
3 & 5 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
\end{array}
\]
Comparing 2 k-bit words

\[ \Theta(k) \text{ steps to compute} \]
\[ \text{Sort in } \Theta(Nk) \text{ time} \]

Faster comparison - binary tree

\[ \Theta(\lg k) \text{ compare} \]
\[ \text{Sort in } \Theta(N\lg k) \]
\[ \lg = \log_2 \]
Pipelining

- compare while sorting
- stagger bits of input

Each processor:

\[
0/1 \rightarrow 0/1 \rightarrow 0/1
\]

Time = \( \Theta(N+k) \) bit steps. Can we do better on \( N \times k \) array?

Lower bounds

1. I/O bandwidth
   \( Nk \) bits to input at \( k \) places \( \Rightarrow \Omega(N) \) steps.

2. Network diameter
   \( \Omega(N+k) \)

3. Communication bandwidth (bisection width)
   \[
   T \geq \frac{\# \text{bits crossing cut}}{\text{size of cut}}
   \]
   \[
   T \geq \frac{\Theta(Nk)}{\Theta(k)} = \Theta(N).
   \]
Problem. $N$ 1-bit #'s input at $N$ leaves of complete binary tree. Time to sort?

$I/O$: $T \geq N/N = 1$

Diam: $T \geq 2 \log N$

$BW$: $T \geq \Theta(N)/1 = \Theta(N)$

:. Sorting $N$ 1-bit #'s takes $\Omega(N)$ time on CBT.

Wrong! Can sort in $O(\log N)$ time!

Idea: Only need to count # 0's.

Input: 1 0 1 1 0 0 1 1
Output: 0 0 0 1 1 1 1

Sum 0's upward, select downward.

1-bit summer

$\text{LSB}(x+y+z) = \text{parity}$

$\text{MSB}(x+y+z) = \text{majority}$

Q. Why doesn't BW lower bound hold?
A. Didn't really need to ship $\Theta(N)$ bits across bisection. Could encode into more compactly. Careful!