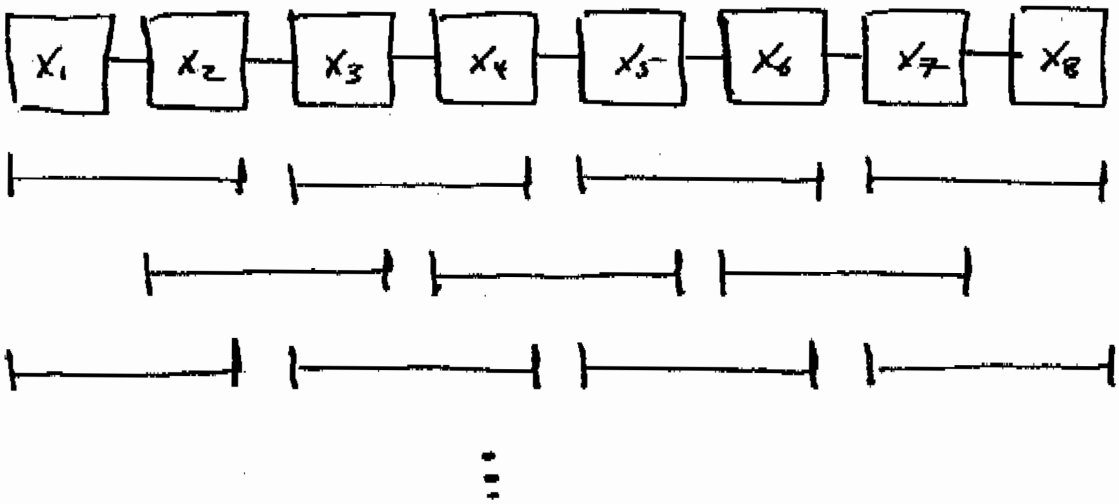
Mon 3/15/04
Michael Bender Lecturing

Sorting on 1 and 2D Arrays

Linear array: odd-even transposition sort.



~~Thm: odd-even trans sort runs in N steps (with 1 of OPT).~~

<u>Def:</u> Oblivious comparison-exchange alg. Comparisons prespecified. Independent of results of prev comparisons.

<<eg. quicksort <u>NOT</u> oblivious>>

<u>Thm:</u> If an obliv comparison-exchange alg sorts all 2^N sequences of 0's and 1's, it sorts all sequences of arbitrary #s

3/15/04

Proof: In 2 parts:

(1) Let f be monotonically increasing function.

Then $\min\{f(x), f(y)\} = f(\min\{x,y\})$

$\max\{f(x), f(y)\} = f(\max\{x,y\})$.

By induction on timesteps, if alg transforms

$$\langle a_1, x_2, \ldots, a_n\rangle \rightarrow \langle b_1, b_2, \ldots, b_n\rangle,$$

then it transforms

$$\langle f(a_1), f(a_2), \ldots, f(a_n)\rangle \rightarrow \langle f(b_1), f(b_2), \ldots, f(b_n)\rangle$$

<<see CLR>>

(2) Suppose false. Ie, network sorts all 0-1 seq, but $\exists \langle a_1, a_2, \ldots, a_n\rangle$ st $a_i < a_j$, but a_i comes after a_j in output.

Define $f(x) = \begin{cases} 0, & \text{if } x \le a_i \\ 1, & \text{if } x > a_i \end{cases}$.

But network fails to sort $\langle f(a_1), f(a_2), \ldots, f(a_n)\rangle$

Contradiction. $\blacksquare$

<<threshold intuition>>

$\Longrightarrow$ Need only construct 0-1 sorting algs!

3/15/04

<u>Thm:</u> Odd-even transposition sort runs in N steps (with 1 of OPT).

<< Result less interesting than proof method >>

<u>Pf:</u> Consider movement of rightmost 1.

1^{st} step: may not move $\underline{0\ 1}$

During subsequent steps, moves forward.

$\Rightarrow$ cannot block other 1s.

$\Rightarrow$ K^{th} leftmost 1 begins moving by step $k+1$. Must reach position $N-k+1$.

$\Rightarrow$ All elmts in final position by time N

3/15/04

Sorting on 2D Grid

Lower bounds: $2\sqrt{N}-2$ (diameter)
$\Omega(\sqrt{N})$ (bisection)

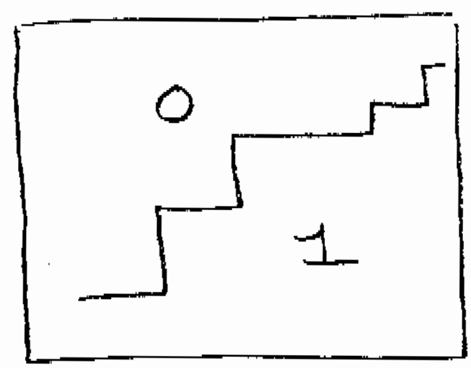
Natural Grid Orders:

"Broken" Alg:

Repeat:

(1) ↓↓↓↓

(2) ⇉



Doesn't yield unique order:

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \Rightarrow \begin{cases} \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} & \downarrow\downarrow \rightrightarrows \\ \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} & \rightrightarrows \downarrow\downarrow \end{cases}$$

3/15/04

Shearsort

Repeat

- $\downarrow\downarrow\downarrow\downarrow$
- $\longrightarrow$
$\longleftarrow$
$\longrightarrow$
$\longleftarrow$

Thm: Shearsort produces unique sorting order after time $O(\sqrt{N}\lg N)$. Eg, $O(\lg N)$ phases sufficient to sort.

Pf: Apply 0-1 lemma.

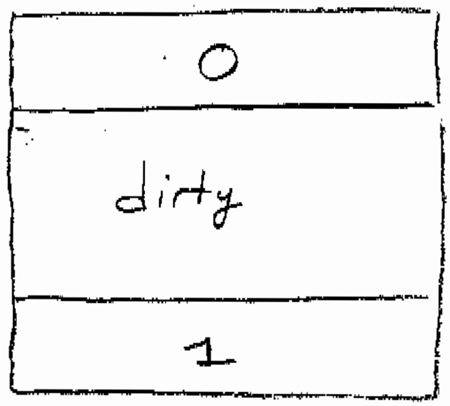
Def:

$\left.\begin{array}{l} 00 \text{———} 00 \\ \quad\text{or} \\ 11 \text{———} 11 \end{array}\right\}$ "clean" lines

$\left.\begin{array}{l} 0 \text{——} 01 \text{——} 1 \\ \quad\text{or} \\ 1 \text{—} 10 \text{——} 0 \end{array}\right\}$ "dirty" lines

Claim: After each phase, # dirty rows decreases by at least half.

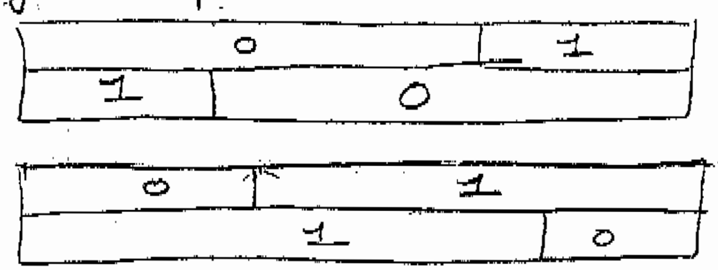
Grid has 3 regions:



Divide dirty into pairs of rows:

either:

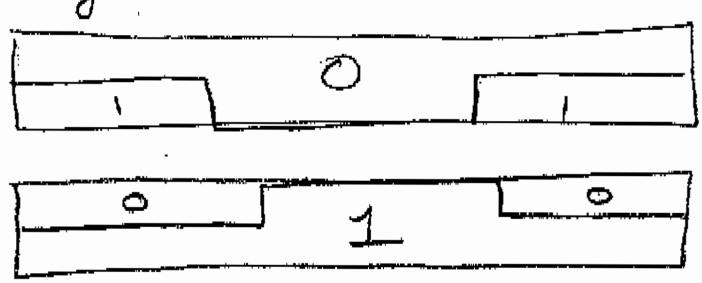
or:



After sorting columns:

either

or

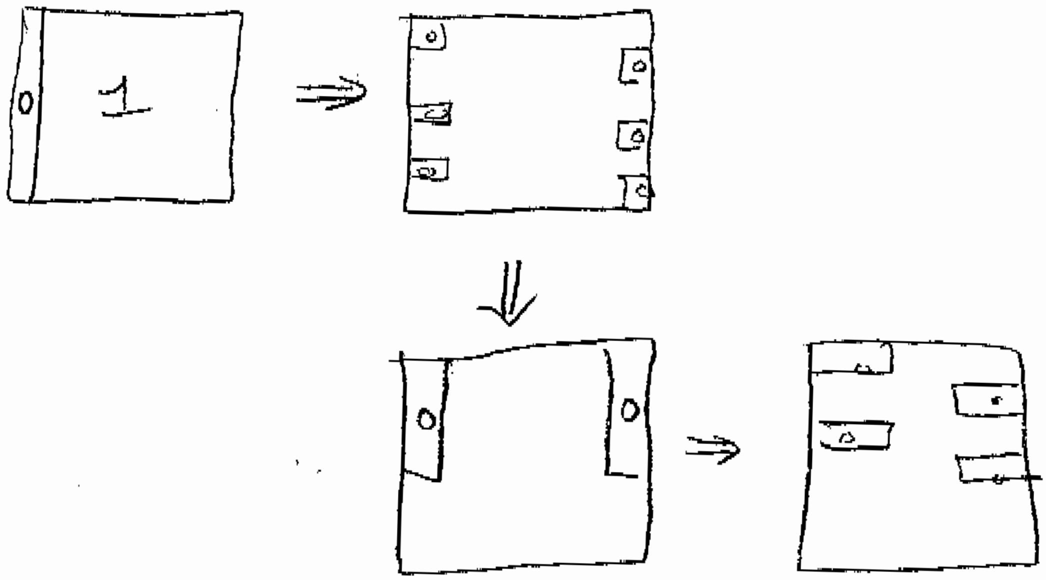


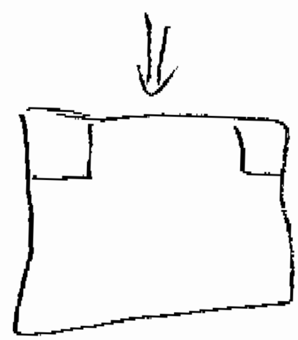
$\Rightarrow$ dirty region decreases by $\geq 1/2$.

$\Rightarrow$ after $\lg N$ phases [$\Theta(\sqrt{N} \lg N)$ time]
all sorted.

Lemma: Shearsort runs in $\Omega(\lg N)$ phases.

Pf: Bad example 0's in 1st column.





height of 1st column decreases by factor of 2 in each round.

Average Case:

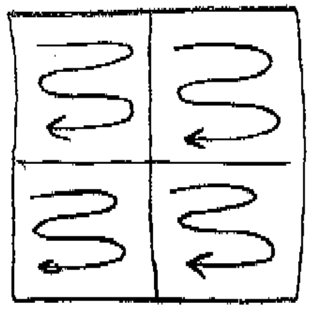
Substitute 0's for $\sqrt{N}$ smallest elmts

- row sort first: $E[\# 0\text{'s in } 1^{st} \text{ column}] = \Theta(N)$.
- column sort first: not true.
  best $LB = \Omega(N \lg\lg N)$.

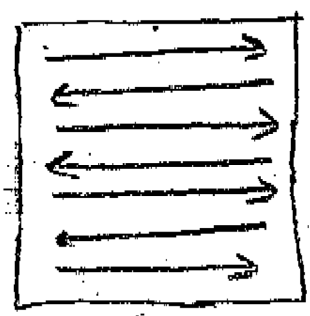
$O(\sqrt{N})$ Algorithm ($\le 8\sqrt{N}$)

Assume $\sqrt{N}$ is power of 2

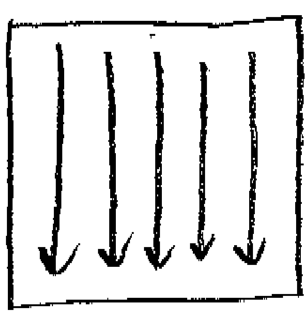
(1) Recursively sort each quadrant in snake order



(2) Sort rows in alternate order



(3) Sort columns



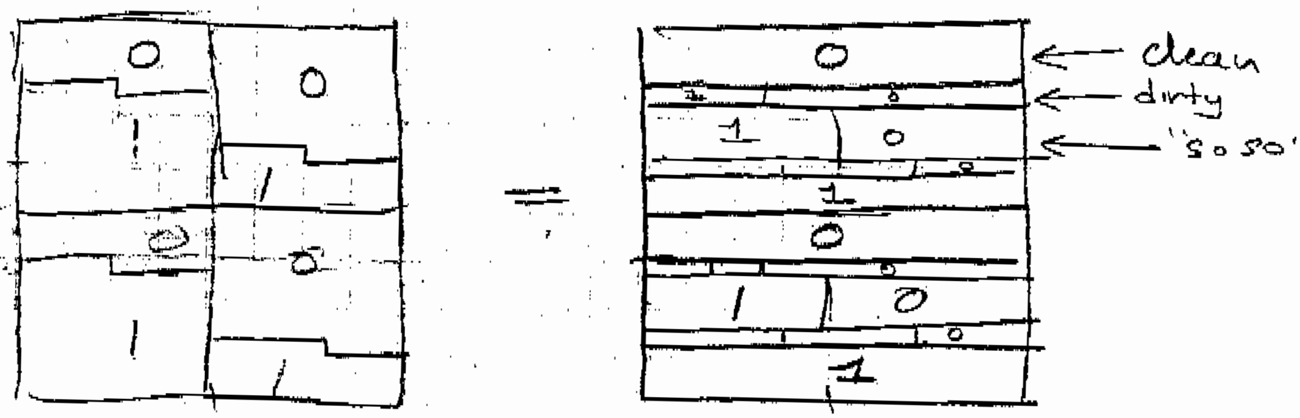
(4) Do $2\sqrt{N}$ steps of 1D odd-even transposition on overall snake order.

Running time:

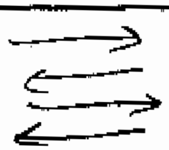
$$T(N) \le T(N/4) + \sqrt{N} + \sqrt{N} + 2\sqrt{N}$$
$$\le 8\sqrt{N}$$

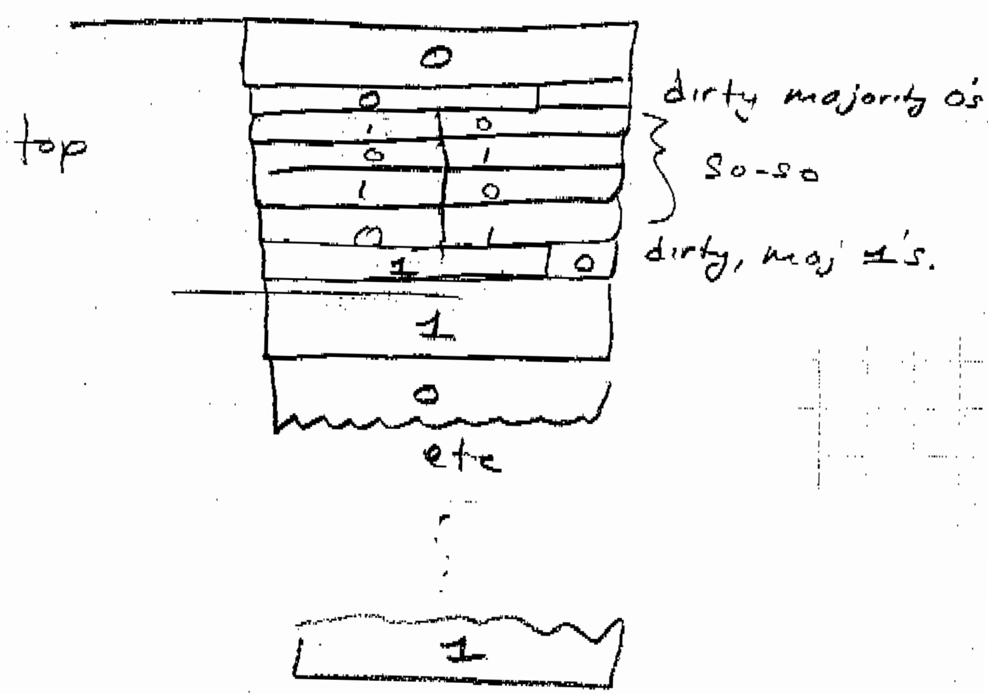
Proof of Correctness:

$\boxed{\text{Phase 1}}$ In each quadrant at most on of rows is dirty and rest are clean:



$\boxed{\text{phase 2}}$



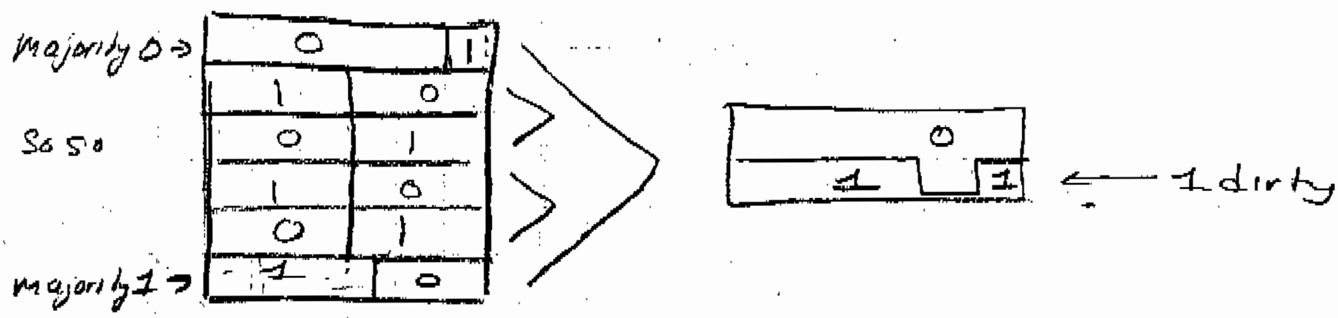


phase 3 ↓↓↓↓

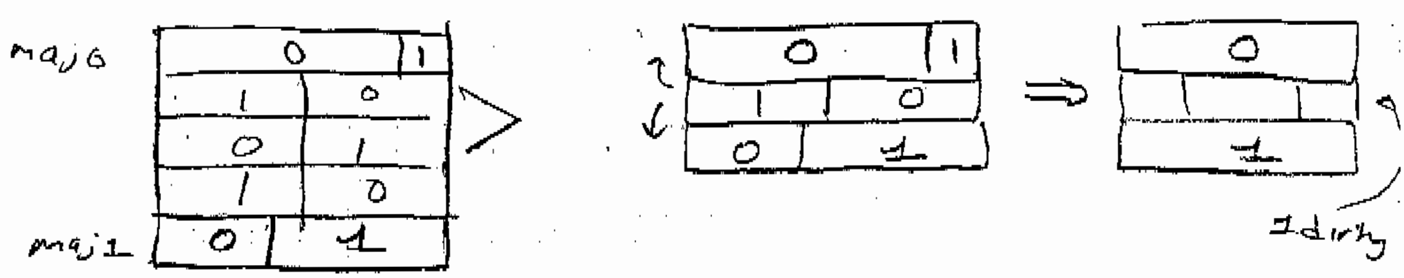
clm: ≤ 1 dirty row in top half &
" " " " bottom half.

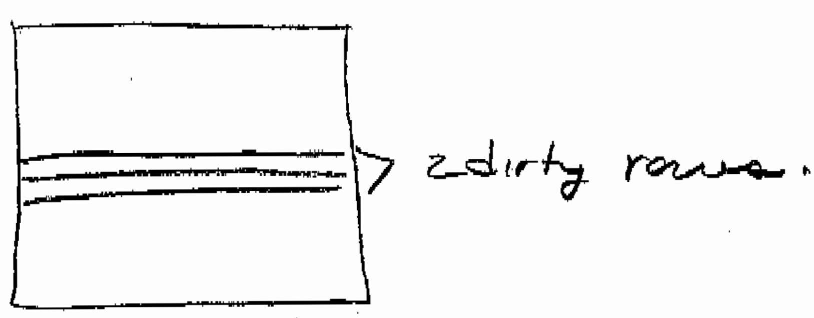
<<gilding the lily - we already have a $\Theta(\sqrt{n})$ alg>>

Pf: Case 1: # so-so rows even:



Case 2: # so-so rows odd:





⇒ Sorted after phase 4!