**Hypercube network**

- **d** dimensions
- \( N = 2^d \) nodes

\[ \begin{align*}
  &d = 0 \\
  &N = 1 \\
  &d = 1 \\
  &N = 2 \\
  &d = 2 \\
  &N = 4 \\
  &d = 3 \\
  &N = 8 \\
  &d = 4 \\
  &N = 16
\end{align*} \]

Label each of the \( 2^d \) nodes with a \( d \)-bit binary string:

\[ b_{d-1} b_{d-2} \ldots b_0 \]

Connect two nodes if they differ in exactly 1 bit:

\[ b_{d-1} b_{d-2} \ldots b_0 \]

\[ \text{connected to} \]

\[ b_{d-1} b_{d-2} \ldots b_0 \]

\[ b_{d-1} b_{d-2} \ldots b_0 \]

\[ \vdots \]

\[ b_{d-1} b_{d-2} \ldots b_0 \]

**Diameter** = \( d = \lg N \)

**Degree** = \( d = \lg N \)

**\( B/W = N/2 \)**

**# wires** = \( Nd/2 = \Theta(N \lg N) \)
Embeddings in the hypercube

Theorem The $N$-node hypercube contains an $N$-node linear array as a subgraph (i.e., a hamiltonian path).

Proof. True for $N=1, 2, 4$.

\[
\begin{array}{c}
0 \quad 0 \\
\end{array}
\] 

Induction on $d$. Claim a hamiltonian cycle for $d$-dim hypercube for $d \geq 2$.

Base:

\[
\begin{array}{c}
\end{array}
\] 

Assume claim true for $N/2$-node hypercube.
Consider $N=2^d$ hypercube.

\[
\begin{array}{c}
\end{array}
\]

Consists of 2 $N/2$-node hypercubes containing (identical) hamiltonian cycles (by IH). Let $(Ox_1, Ox_2)$ be any edge in 1st subcube that cycle goes through, and let $(1x_1, 1x_2)$ be corresponding edge in 2nd subcube. Replace these two edges with $(0x_1, 1x_1)$ and $(Ox_2, 1x_2)$. \( \square \)
Def. A d-bit Gray code is an ordering of the $2^d$ d-bit bit-strings such that each string differs from the previous in exactly one bit.

Ex. $d=3$

```
0 0 0 0
1 0 0 1
2 0 1 1
3 0 1 0
4 1 1 0
5 1 1 1
6 1 0 1
7 1 0 0
```

"Reflecting" Gray code

Corollary. d-bit Gray codes exist if $d \geq 1$.

Hamiltonian path in hypercube = Gray code.

Theorem. Let $d_1 + d_2 \geq d$. Then a $2^{d_1} \times 2^{d_2}$ mesh (or torus) can be embedded in an $N = 2^d$-node hypercube.

Pf. Let $g_1(x_1)$ be $d_1$-bit Gray code of $x_1$, where $0 \leq x_1 < 2^{d_1}$.

Let $g_2(x_2)$ be $d_2$-bit Gray code of $x_2$, where $0 \leq x_2 < 2^{d_2}$.

Map node $(x_1, x_2)$ of mesh to node $g_1(x_1) \parallel g_2(x_2)$ of hypercube.

Ex. $8 \times 8$ mesh.

```
(3, 6) 0 1 0 1 0 1 0 1
(4, 5) 1 1 0 1 1 0 1 0
(4, 6) 1 0 1 0 1 1 0 1
(4, 7) 0 1 0 1 0 1 0 1
(5, 6) 1 1 1 0 0 0 0 0
```

Corollary. $2^{d_1} \times 2^{d_2} \times \ldots \times 2^{d_k}$ mesh embedded in $2^{d_1 + d_2 + \ldots + d_k}$ hypercube.

Fact: $3 \times 5$ mesh cannot be embedded in 16-node hypercube.

But $m \times n$ mesh can be embedded in $2^{\lceil \log_2 mn \rceil}$-node hypercube with dilation 2.
Embedding trees in hypercubes

Thm. Not possible to embed (N-1)-node complete binary tree in N-node hypercube.

Proof. Suppose possible. Root mapped to node 00...0.
Depth-1 nodes mapped to nodes with odd parity.
Depth-2 " " " " even " " odd " " odd (Def. Parity = \{odd if #1's is odd, even if #1's is even\})

#leaves = N/2: all have same parity.
#grandparents of leaves = N/8: same parity as leaves.

But, hypercube has N/2 nodes with even parity and N/2 nodes with odd parity, and tree must have ≥ N/2 + N/8 nodes with same parity.

Def. Double-rooted complete binary tree:

\[ \text{Thm. } N\text{-node double-rooted CBT is subgraph of } N\text{-node hypercube, for } N \geq 2. \]

Proof. Induction on d.
\[ d = 1 \text{ (N=2): } \]
\[ d = 2 \text{ (N=4): } \]
\( d \geq 3 \) \((N \geq 8)\): Embed double-rooted cft on \( N/2 \) nodes in \( N/2 \)-node 0-subcube. Consider top 4 nodes:

\[
\begin{array}{c}
\text{a} & \text{b} & \text{c} & \text{d} \\
0000 & 0010 & 0100 & 0110 \\
\end{array}
\]

\[ a, b \text{ differ in dim } i_0, \quad b, c \text{ differ in dim } i_1, \text{ and } c, d \text{ differ in dim } i_2. \]

Note: \( i_0 \neq i_1 \neq i_2 \) \(\text{or else } a = c \text{ or } b = d\).

Embed double-rooted cft on \( N/2 \) nodes in \( N/2 \)-node 1-subcube identically, except \( b' = 100 \ldots 0 \) and permute dimensions \( i_1 \rightarrow i_0 \) and \( i_2 \rightarrow i_1 \).

Thus, \((a, b'), (b, c'), \) and \((c, d')\) adjacent.
Corollary: $(N-1)$-node CBT embeds in $N$-node hypercube with dilation 2.

**Proof:**

Embed CBT into double-rooted CBT with 1 edge having dilation 2. \(\square\)

Fact: All $N$-node binary trees can be embedded into $N$-node hypercube with $O(1)$ dilation. \(\leq 5\)