Divide-and-Conquer Layout

A general algorithm for creating a good layout for any bounded-degree tree on planar graphs, even near-optimal scale.

A general layout algorithm, near-optimal layout for the graph, optimal for bounded degree trees + planar graphs.

Idea:

Layout (G):
1. Find small number of edges that disconnect the graph into nearly equal pieces. (Can be tough)
2. Recursively layout the two halves
3. Put the two halves together, wrap it up.

Definitions:

- G has an S-separator (i.e., S-separable) if
  - G has 1 vertex, or
  - G has an S-separator (i) S-separable
  - For a set A \subseteq E s.t. |A| < S(|V|)
    and (V, E - A) is two disconnected graphs

\[ G_1, G_2, G_3 = (V_1, E_1) \]
\[ G_4 = (V_2, E_2) \]

s.t. \[ |V_1| \geq \frac{1}{2} \quad \text{and} \quad |V_2| \geq \frac{1}{2} \]
and \[ G_1 + G_3 \] are S-separable.
Example:

Theorem: Binary trees are 1-separated.

Proof: Pick a root.

Travel down the tree looking for a node that is the center of 
\[ \frac{c}{3} \text{ to } \frac{2c}{3} \text{ nodes.} \]

That makes the edge disjoint from the tree.

- Case A: The subtree is like \( \frac{c}{3} + \frac{2c}{3} \) nodes.

- Case B: Subtree too small. We do not know if there are two big subtrees. > \( \frac{2c}{3} \) nodes.

One of two children is at least half \( \frac{2c}{3} \) nodes, 
so go to plus part.

- Case C: Too small. We don't go there.

Example:

1. 19 nodes, 1 node, too big
2. 17 nodes, too big
3. 15 nodes, 13 nodes, too big
4. 10 nodes, 12 nodes, OK

Top half is 8 nodes 
geofor... into 5 + 5

Top 1/4 an 5 nodes

uthe... and so forth

SAVE THIS TREE
Defn: A pushdown tree of a G.P.A.A. is a tree where each node is defined by example:

\[ \{4, 5, 93\} \quad \{1, \ldots, 19\} = \{1, 19\} \]

\[ \{1, 7, 93\} \]

\[ \{1, 3, 5, 93\} \quad \{4, 6, 7\} \]

\[ \{1, 3\} \quad \{2, 5, 93\} \]

\[ \{13, \{22, 52\}\} \quad \{5, 93\} \]

\[ \{5, 19\} \]

This tree is a tree even if G is not a tree.
Example: A grid is $O(\sqrt{n})$ separable.

Defn: $G$ has a strong $S$-separator if the sizes of the subsets are at most $\frac{(n+1)}{2}$. [cut exactly in half or, otherwise as close as possible]

Claim: not interesting. We'll show $S$ is $\Omega(n^c)$ implies $S$-separable or strongly separable.

Defn: $\Gamma$ (gamma) defined as

$$\Gamma_n = S(n) + S\left(\frac{2}{3}n\right) + S\left(\frac{4}{7}n\right) + \ldots$$

$$= \sum_{i=0}^{\infty} S\left(\frac{2^i}{3^i}n\right)$$

Ex: $S(n) = n^d \log n$

$$\Gamma_n = n^d \cdot \left(\frac{2}{3}\right)^0 \cdot \left(\frac{4}{7}\right)^1 \cdot \ldots$$

$$= n^d \cdot \left(1 + \frac{2}{3} + \frac{4}{7} + \ldots \right) \approx n^d \cdot \frac{1}{1 - \left(\frac{2}{3}\right)} = O(n^d)$$
proof by induction: \( p(n) \leq \sqrt{n} \)

if \( n \leq t \) then add to left subtree to our selected set + go right to \( p(n) = S(n) + \sqrt{\frac{n}{2}} \).

if \( n > t \) then go left

claim: the edge connecting a selected set to another set \( \leq S(n) \) edges connecting selected sets to anything else.

pf.

if \( \epsilon \) is selected in a set, connect \( u \) to non-selected

if \( \epsilon \) is not selected in any set, connect \( u \) to the selected set.

if \( u \) is selected, then at most \( S(n) \) of its edges to \( u \) are needed to connect \( u \) to non-selected nodes.

if \( u \) is not selected, then at most \( S(n) \) of those edges are needed to connect \( u \) to the selected set.

The total number is then \( 2 \sum S(n) = S(n) + \frac{S(n)}{2} + \frac{S(n)}{4} \) = \( \sum S(n) \).
Example: 

Bin tree are 1-separated \( \Rightarrow \) they are strongly, log-separated.

Back into 10 nodes,
Select 10 nodes

Select top 6, if + need to select 2 nodes

--- is too big, select next

\( \Pi_1(19) = \Pi_{13}, \lambda_0 = 7 = 8 \)  

so we did ok.
6.896
18.8
4-21-04

Back to layout algorithm
1) Separate
2) Recise
3) Reassemble

cut in 1/3 with space
did expensive layout or help now what?

need to correct two oles.

Q: There is a tree
Most have fun according least

in a chord stretch right vertically to ensure you leave a space between farthest notes
Example: \( S(n) = O(1) \) for
\[ \Gamma_G(n) = O(\log n) \]

Example: \( S(n) = \log n \) for
\[ \Gamma_G(n) = \log n + \log \frac{2}{3} n + \log \frac{4}{3} n + \ldots \]
\[ = \log n + \Gamma_G(\frac{2}{3} n) \]
\[ = \log^2 n \]

Lemma: If \( G \) is \( S \)-separable then \( G \) is strongly \( \Gamma_G \) separable.

Proof: for any \( t \leq |V| \) a path is the partition tree

Proof: Build a partition tree for \( G \) achieves \( S \)-separation

For any \( t \leq |V| \) we will find a collection such that:
- the path from the root of the partition tree to a leaf, & some subset of the siblings add up to exactly \( t \) nodes.

\[ \text{E.g.,} \]

- Here is the \( t \), \( 2 \), \( 3 \).
- Pick two next ready t
- Call three for selected sets
Examples: layout as a line array.

\[ \begin{array}{c}
\text{two vertical bars} \\
\text{stretch vertically to bring} \\
\text{stretch horizontally between two bars}
\end{array} \]

Defn. \( \Delta_s(n) = s(n) + 2s(\lfloor n/4 \rfloor) + s(n/16) \ldots \)

\[ \begin{align*}
\Delta_s(n) &= s(n) + 2\Delta_s(\lfloor n/4 \rfloor) \\
\Delta_s(n) &= n^d + 2(n/4)^d + 4(n/16)^d = \Theta(n^d) \\
\end{align*} \]

\[ \begin{cases}
\Delta_n^\alpha(n) = O(n^{\alpha}) & \text{if } \alpha < \frac{1}{d} \\
\Delta_n^\alpha(n) = O(n^{\alpha}) & \text{if } \alpha \geq \frac{1}{d} \\
\Delta_n^\alpha(n) = O(n^{\alpha} \log n) & \text{if } \alpha > \frac{1}{d}
\end{cases} \]

\[ \begin{align*}
\text{if } \alpha < \frac{1}{d} \text{ then } & \quad \Delta_s(n) = O((\log n)^d) \\
\text{if } \alpha > \frac{1}{d} \text{ then } & \quad \Delta_s(n) = O(n^d) \\
\text{if } \alpha = \frac{1}{d} \text{ then } & \quad \Delta_s(n) = \Delta(n) = O((\log n)^d)
\end{align*} \]
Then: $S(n)$ monotonically non-decreasing.

A sort of with a strong $S$-separator can be placed in a square with side length $O\left(\max(\sqrt{n}, D_s(n))\right)$

Induction on $n$, assume $n$ a power of 4

Claim side length $\geq \sqrt{n} + 6 D_s(n)$ good enough

Base case: easy to see

Inductive: divide a half + half case

\[ W(n) = 2\left(\frac{\sqrt{n}}{3} + 6 D_s\left(\frac{n}{4}\right)\right) + S(n) \]

but $D_s(n) = S(n) + 2D_s\left(\frac{n}{4}\right)$

\[ = \sqrt{n} + 6 D_s\left(\frac{n}{4}\right) \]

\[ N(n) = S(n) + S\left(\frac{n}{2}\right) + 2H\left(\frac{n}{4}\right) = O(S(n)) + H\left(\frac{n}{4}\right) \]

Analysis: $S(n) = O(n^\alpha)$ for $\alpha < \frac{1}{3}$: $N(n) = O(\sqrt{n}) + H(\sqrt{n}) = O\left(\sqrt{n} + \frac{\sqrt{n}}{3} + \sqrt{\sqrt{n}}\right)$