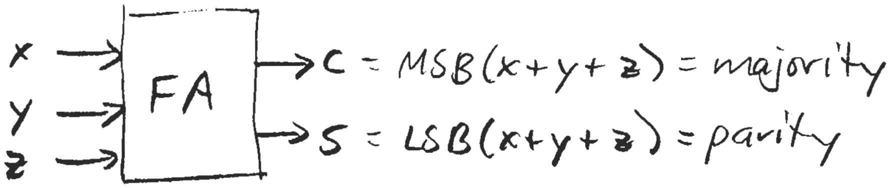
<< Sort with $\Theta(\lg N)$ procs in linear array>>
<< Firing squad – $\Theta(1)$-size state>>

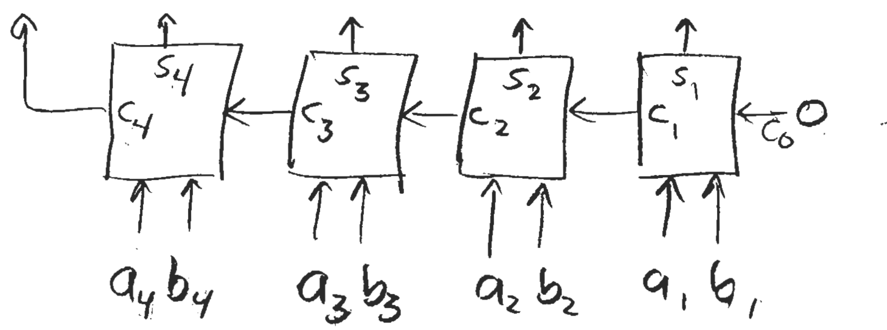
Addition

Basic component: Full adder – combinational



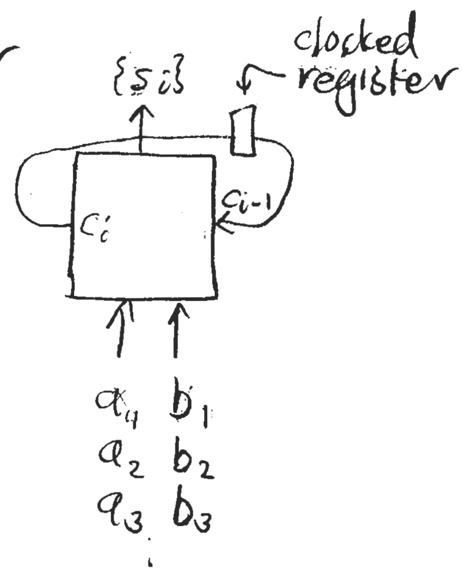
Problem:
Add 2 N-bit numbers

Ripple-carry adder



N-bit #'s ⇒ $\Theta(N)$ time, $\Theta(N)$ HW, combinational

Serial adder



$\Theta(N)$ time, $\Theta(1)$ hardware, sequential (clocked)

Fast addition

Idea: carries are the hard part.
Know carries ⇒ compute sum in $\Theta(1)$ time
How? Array of full adders. «Show on ripple-carry adder»

	1	0	1	0	1	0	1
	1	1	1	0	0	1	1
1	1	0	0	1	0	0	0
	g	p	g	k	p	p	g (k)

Classify stages:

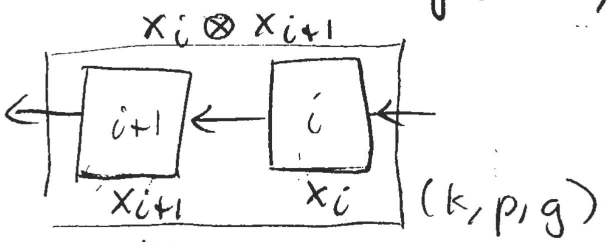
kill: $\begin{smallmatrix}0\\0\end{smallmatrix}$ ⇒ carry-out = 0

propagate: $\begin{smallmatrix}0\\1\end{smallmatrix}$ or $\begin{smallmatrix}1\\0\end{smallmatrix}$ ⇒ carry-out = carry-in

generate: $\begin{smallmatrix}1\\1\end{smallmatrix}$ ⇒ carry-out = 1

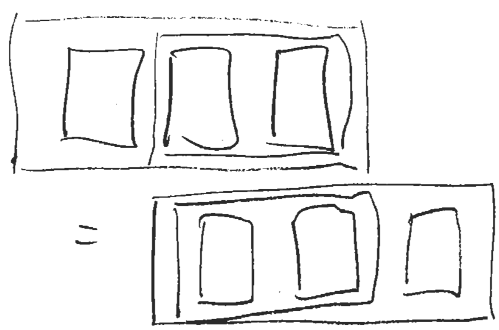
Carry into stage $= \begin{cases} 1 & \text{if most recent non-p is k} \\ 0 & \text{otherwise} \end{cases}$

When do 2 consecutive stages kill, prop, gen?



$\otimes$ (x_i \ x_{i+1})	k	p	g
k	k	k	g
p	k	p	g
g	k	g	g

Associative!



Theorem. Let x_i = carry status of stage i, where $x_0 = k$. Define $y_i = x_0 \otimes x_1 \otimes \cdots \otimes x_i$.

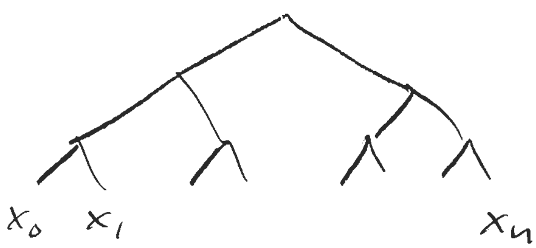
Then
$$y_i = k \Rightarrow c_i = 0$$
$$y_i = g \Rightarrow c_i = 1$$
$y_i = p$ does not occur.

Proof. Induction on i. ⊠

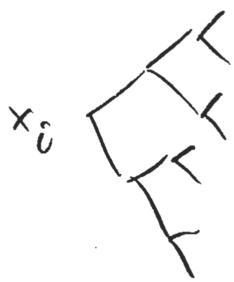
Log-time circuit:

$$y_0 = x_0$$
$$y_1 = x_0 \otimes x_1$$
$$y_2 = x_0 \otimes x_1 \otimes x_2$$
$$y_N = x_0 \otimes x_1 \otimes \cdots \otimes x_N$$

Use tree for each calculation:



Use tree to broadcast inputs (bounded-degree network):



Time = $\Theta(\lg N)$, HW = $\Theta(N^2)$.

Carry-lookahead addition

$\Theta(\lg N)$ time, $\Theta(N)$ HW.

"Parallel prefix"

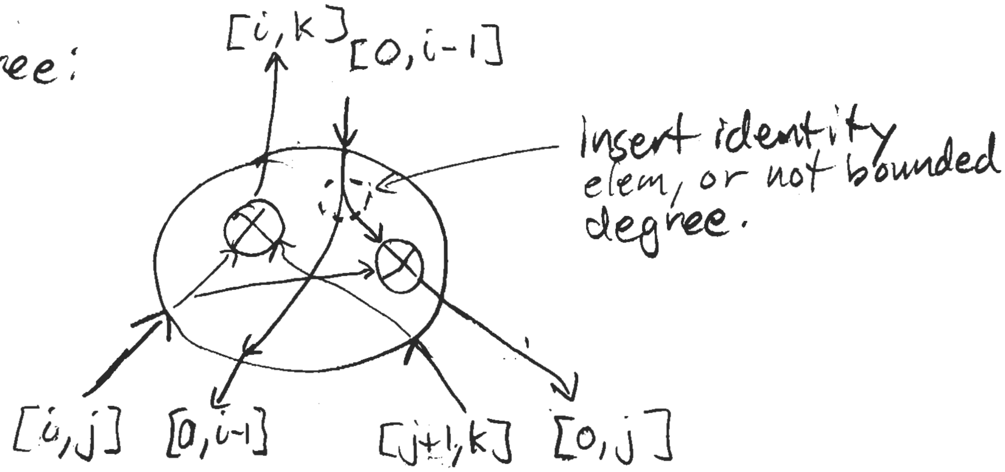
Let $[i, j]$ denote $x_i \otimes x_{i+1} \otimes \cdots \otimes x_j$

Lemma. $[i, j] \otimes [j+1, k] = [i, k]$ ⊠

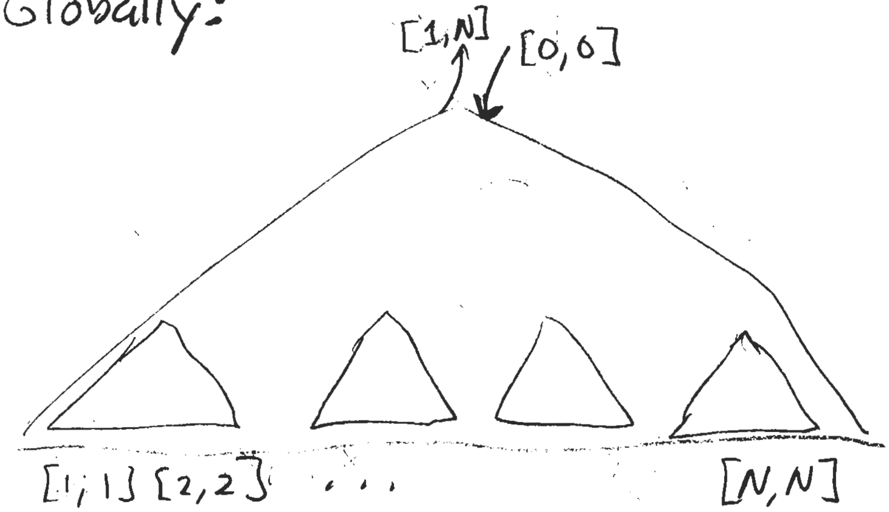
$x_i = [i, i]$

$y_i = [0, i]$

Build tree:

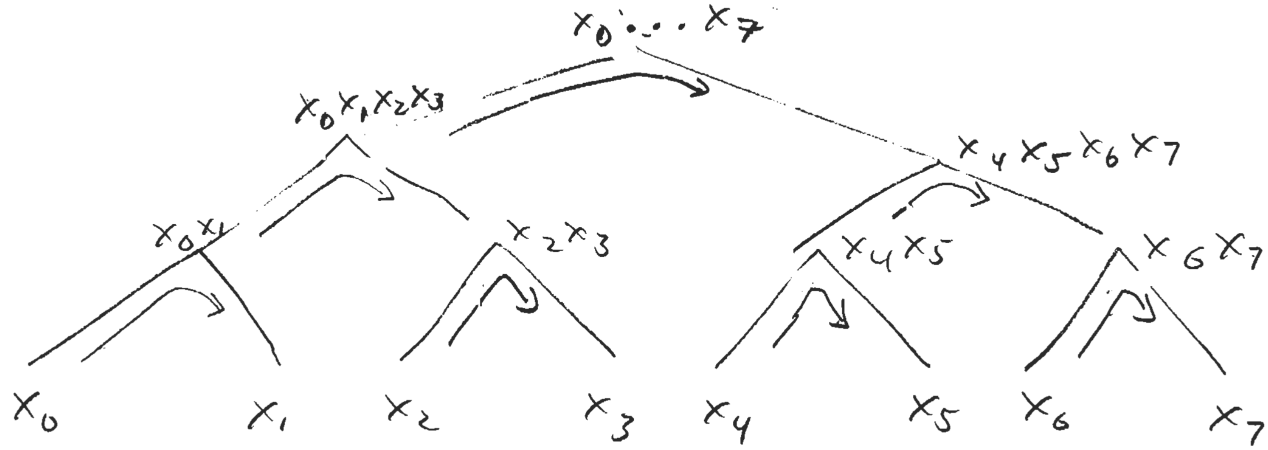


Globally:

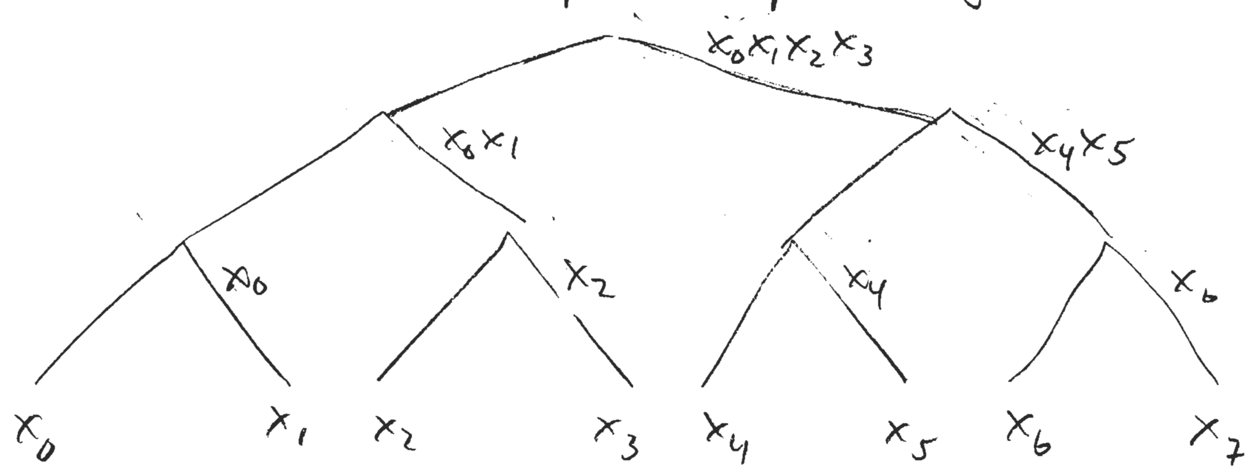


Left child values are passed up.

Similar method:



Left child values are passed up and right



<u>Postscript</u> Kill, propagate, generate first used in Harvard relay calculator circa mid-1940's.

O(1)-time addition (in their model).