MIT 6.972
Algebraic methods and semidefinite programming
Homework assignment # 3

Date Given: May 12th, 2006
Date Due: May 19th, 12PM

P1. [30 pts] Recall the relaxations for linearly and quadratically constrained quadratic programming we have seen earlier (concretely, equation (9) in Lecture 3). Explain how these can be interpreted as a special case of Positivstellensatz-based relaxations (and more specifically, a Schm¨udgen-type certificate).

P2. [35 pts] In this problem, we analyze symmetry reduction in the case of sum of squares decompositions of univariate even polynomials. Let $p(x)$ be a univariate polynomial that satisfies $p(x) = p(-x)$ (i.e., it is even).

(a) Write down the “standard” SDP formulation for checking whether $p(x)$ is SOS.

(b) Is this SDP invariant under the action of a group?

(c) Restrict the feasible set to the fixed-point subspace. How does the problem simplify? (Hint: you may want to group the monomials depending on whether the exponents are even or odd).

(d) Explain why the new formulation is computationally better.

(e) Compare the results with making the substitution $y = x^2$ in the original polynomial, and imposing the constraint $y \geq 0$. How do they differ (if they do)?

P3. [35 pts] Let $M \in S^n$, and let $z = [x_1^2, \ldots, x_n^2]^T$. As we have seen, $M$ is copositive if and only if the homogeneous quartic polynomial $p(x) = z^T M z$ is nonnegative.

(a) Plot the region of $(a, b) \in \mathbb{R}^2$ for which the matrix

$$
\begin{bmatrix}
a & b \\
b & 1 \\
\end{bmatrix}
$$

is copositive.

(b) Prove that $p(x)$ is a sum of squares if and only if $M = P + N$, where $P$ is positive semidefinite and $N$ is componentwise nonnegative. (Hint: you may want to use the symmetry $p(x_1, \ldots, x_n) = p(\pm x_1, \ldots, \pm x_n)$).