Single Period Inventory Models

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Outline

- Single period inventory decisions
- Calculating the optimal order size
  - Numerically
    - Using spreadsheet
    - Using simulation
  - Analytically
- The profit function
  - For specific distributions
- Level of Service
- Extensions:
  - Fixed costs
  - Risks
  - Initial inventory
  - Elastic demand
Single Period Ordering
Selling Magazines

- **Weekly demand:**

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 | 48 | 87 | 78 | 58 | 71 | 102 | 87 | 66 | 79 | 97 | 75 | 89 | 90 |
| 57 | 86 | 95 | 67 | 89 | 70 | 113 | 52 | 84 | 62 | 91 | 71 | 66 | 99 | 73 | 92 | 66 | 67 | 89 | 87 | 64 | 70 | 54 | 67 | 88 | 62 | 79 | 79 | 105 | 76 | 73 | 78 | 50 | 107 | 80 | 78 | 51 | 79 | 80 |

- **Total:** 4023 magazines
- **Average:** 77.4 Mag/week
- **Min:** 51; **max:** 113 Mag/week
Detailed Histogram

Average = 77.4 Mag/wk
Histogram

Histogram of Demand (Mag/week) vs Frequency (Wks/Yr)

Cumulative Frequency
# The Ordering Decision (Spreadsheet)

- Assume: each magazine sells for: $15
- Cost of each magazine: $8

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</table>
Expected Profits

![Graph showing expected profits versus order size](image-url)
Optimal Order  (Analytical)

- The optimal order is \( Q^* \)
- At \( Q^* \) - what is the probability of selling one more magazine?
- The expected profit from ordering the \((Q^*+1)\)st magazine is:

\[
\text{If demand is high and we sell it:} \quad (\text{REV-COST}) \times \text{Pr( Demand is higher than } Q^*)
\]
\[
\text{If demand is low and we are stuck:} \quad (-\text{COST}) \times \text{Pr( Demand } \leq Q^*)
\]

The optimum is where the total expected profit from ordering one more magazine is zero:

\[
(\text{REV-COST}) \times \text{Pr( Demand } > Q^*) - \text{COST} \times \text{Pr( Demand } \leq Q^*) = 0
\]

\[
\text{Pr( Demand } \leq Q^*) = \frac{\text{REV-COST}}{\text{REV}}
\]
Optimal Order

The "critical ratio": $Pr(\text{Demand} \leq Q^*) = \frac{\text{REV-COST}}{\text{REV}} = \frac{15 - 8}{15} = 0.47$
Salvage Value

Salvage value = $4/Mag.  Critical Ratio = \( \frac{REV - COST}{REV - SLV} = \frac{15 - 8}{15 - 4} = 0.64 \)
The Profit Function

- Revenue from sold items
- Revenue or costs associated with unsold items. These may include revenue from salvage or cost associated with disposal.
- Costs associated with not meeting customers’ demand. The lost sales cost can include lost of good will and actual penalties for low service.
- The cost of buying the merchandise in the first place.
The Profit Function

\[ E[Sales] = Q \cdot \int_{X=Q}^{\infty} f(x)dx + \int_{x=0}^{Q} x \cdot f(x)dx \]

\[ E[Unsold] = \int_{x=0}^{Q} (Q - x) \cdot f(x)dx = Q - E[Sales] \]

\[ E[Lost Sales] = \int_{X=Q}^{\infty} (x - Q) \cdot f(x)dx = \mu - E[Sales] \]

\[ E[Profit] = R \cdot E[Sales] + S \cdot E[Unsold] - L \cdot E[Lost Sales] - C \cdot Q \]
The Profit Function – Simple Case

\[ E[\text{Profit}] = R \cdot E[\text{Sales}] - C \cdot Q \]

Optimal Order:

\[
\frac{d}{dQ} E[\text{Profit}] = (1 - F(Q)) \cdot R - C = 0
\]

\[
\frac{d}{dQ} E[\text{Sales}] = 1 - F(Q)
\]

\[
F[Q^*] = \frac{R - C}{R} \quad \text{and:} \quad Q^* = F^{-1}\left[\frac{R - C}{R}\right]
\]
Level of Service

- Cycle Service – The probability that there will be a stock-out during a cycle
- Fill Rate - The probability that a specific customer will encounter a stock-out
Level of Service

REV = $15
COST = $8
Normal Distribution of Demand

\[ X \sim N(\mu, \sigma) \]

\[ E[\text{sales}] = Q - \sigma \cdot (z \cdot \Phi(z) + \phi(z)) \]

\[ z = \frac{Q - \mu}{\sigma} \]

\[ E[\text{Profit}] = (R - C) \cdot Q - R \cdot \sigma \cdot \left[ z \cdot \Phi(z) + \phi(z) \right] \]

\[ Q^* = \text{NORMINV}\left(\frac{R - C}{R}\right) = \text{NORMINV}\left(\frac{15 - 8}{15}\right) = 76 \text{ Mags} \]
Incorporating Fixed Costs

With fixed costs of $300/order:

REV=$15
COST=$8
Risk of Loss

Order Quantity

Probability of Loss

REV=$15
COST=$8

F=$300
F=0
Ordering with Initial Inventory

Given initial Inventory: $Q_0$, how to order?

- Cost of initial inventory

With fixed costs, order only if the expected profits from ordering are more than the ordering costs
Ordering with Fixed Costs and Initial Inventory

Example: $F = 150$

- If initial inventory is LE 46, order up to 80
- If initial inventory is GE 47, order nothing
Elastic Demand

\[ \mu = D(P); \quad \sigma = f(\mu) \]

Procedure:

1. Set \( P \)
2. Calculate \( \mu \)
3. Calculate \( \sigma \)
4. \( Q^* = F^{-1}\left(\frac{P - C}{P}\right) \)
5. Calculate optimal expected profits as a function of \( P \).

\[
\begin{align*}
\text{Rev} &= $15 \\
\text{Cost} &= $8 \\
\mu(p) &= 165 - 5p \\
\sigma &= \mu/2 \\
\end{align*}
\]

\[
\begin{align*}
P^* &= $22 \\
Q^* &= 65 \text{ Mag} \\
\mu(p) &= 56 \text{ Mag} \\
\sigma &= 28 \\
\text{Exp. Profit} &= $543
\end{align*}
\]
Elastic Demand: Numerical Optimization

Screenshots removed due to copyright restrictions.
Any Questions?

Yossi Sheffi