Inventory Management
Extensions to EOQ

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ESD.260/15.770/1.260 Logistics Systems
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Agenda

- Review of Basic EOQ
- Non-instantaneous Leadtime
- Finite Replenishment (EPQ)
- Multiple Locations
- Discounting
Economic Order Quantity (EOQ)

Finding the order quantity $Q$ (and frequency $T$) that minimizes the total relevant cost.

\[
TRC[Q] = \frac{AD}{Q} + \frac{vrQ}{2}
\]

\[
Q^* = \sqrt{\frac{2AD}{vr}} \quad T^* = \sqrt{\frac{2A}{Dvr}}
\]

\[
TRC^* = \sqrt{2ADvr}
\]
Assumptions: Basic EOQ Model

- **Demand**
  - Constant vs Variable
  - Known vs Random
  - Continuous vs Discrete

- **Lead time**
  - Instantaneous
  - Constant or Variable (deterministic/stochastic)

- **Dependence of items**
  - Independent
  - Correlated
  - Indentured

- **Review Time**
  - Continuous
  - Periodic

- **Number of Echelons**
  - One
  - Multi (>1)

- **Capacity / Resources**
  - Unlimited
  - Limited / Constrained

- **Discounts**
  - None
  - All Units or Incremental

- **Excess Demand (Shortages)**
  - None
  - All orders are backordered
  - Lost orders
  - Substitution

- **Perishability**
  - None
  - Uniform with time

- **Planning Horizon**
  - Single Period
  - Finite Period
  - Infinite

- **Number of Items**
  - One
  - Many

- **Form of Product**
  - Single Stage
  - Multi-Stage
Extensions: Leadtime

- **Order Leadtime**
  - Positive (nonzero)
  - Deterministic

- **Impact**
  - Does this change Q*?
  - What is my new policy?
  - What is my new avg IOH?

**EOQ Inventory Policy:**
Order Q* units when IOH = DL

L = Order Leadtime
Inventory On Order
Extensions: Finite Replenishment

Inventory becomes available at a rate of \( m \) units/time rather than all at one time
- Does this change \( Q^* \)?
- What is my new policy?
- What is my new avg IOH?

\[
\begin{align*}
Q &= Q(1 - D/m) \\
\text{Slope } &= m - D \\
\text{Slope } &= -D \\
\end{align*}
\]

\[
\begin{align*}
TRC[Q] &= \frac{AD}{Q} + \frac{Q}{2} \left(1 - \frac{D}{m}\right)vr \\
EPQ &= \sqrt{\frac{2AD}{vr \left(1 - \frac{D}{m}\right)}} = EOQ \left(\sqrt{1 - \frac{D}{m}}\right)^{-1}
\end{align*}
\]
Extensions: Multiple Locations

Suppose that instead of one location satisfying all demand, there are \( n \) locations.

- Each location serves \( d_i = D/n \) units of demand
- Identical (uniform) demand at each location

Questions

- What is my new inventory policy?
- What is my new average Inventory on Hand?
- How much is this better or worse than a single location?

\[
Q^* = \sqrt{\frac{2AD}{vr}}
\]

\[
\overline{IOH} = \frac{Q^*}{2} \quad TRC^* = \sqrt{2DAvr}
\]

\[
q_i^* = \sqrt{\frac{2Ad_i}{vr}} = \sqrt{\frac{2AD}{v rn}}
\]

\[
\overline{IOH} = \sum_{i=1}^{n} \left( \frac{q_i^*}{2} \right) = \sqrt{n} \left( \frac{Q^*}{2} \right)
\]

\[
TRC^* = \sqrt{2nDAvr}
\]
Extensions: Multiple Locations

- What if I reduce number of stocking locations from M to N?
  \[
  \frac{TRC^*[M]}{TRC^*[N]} = \frac{\sqrt{2MDA_{vr}}}{\sqrt{2ND_{vr}}} = \sqrt{\frac{M}{N}}
  \]

- What if my sub-regions do not have uniform demand?

- Is this a reduction in safety stock, cycle stock, or both?

- How dependent is this effect on inventory policy at each site?
  - EOQ Policy (order \( q_{EOQ^*} \) when \( IOH_i = 0 \))
  - Fixed Order Size (Always order a full truckload at a time)
  - Days of Supply (Always order a month’s supply)
Extensions: Multiple Locations

For a Single Location

<table>
<thead>
<tr>
<th>Policy</th>
<th>EOQ</th>
<th>FOS</th>
<th>DOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order Size</td>
<td>$Q^*$</td>
<td>$Q_{FOS}$</td>
<td>$Q_{DOS}$</td>
</tr>
<tr>
<td>Average IOH</td>
<td>$Q^*/2$</td>
<td>$Q_{FOS}/2$</td>
<td>$Q_{DOS}/2$</td>
</tr>
<tr>
<td>Order Cost</td>
<td>$O_{EOQ}$</td>
<td>$O_{FOS}$</td>
<td>$O_{DOS}$</td>
</tr>
<tr>
<td>Holding Cost</td>
<td>$H_{EOQ}$</td>
<td>$H_{FOS}$</td>
<td>$H_{DOS}$</td>
</tr>
<tr>
<td>Total Cost</td>
<td>$O_{EOQ} + H_{EOQ}$</td>
<td>$O_{FOS} + H_{FOS}$</td>
<td>$O_{DOS} + H_{DOS}$</td>
</tr>
</tbody>
</table>

Example

<table>
<thead>
<tr>
<th>DATA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A =$ 500 $/order</td>
</tr>
<tr>
<td>$D =$ 2000 Units/year</td>
</tr>
<tr>
<td>$r =$ 0.25 $/$/year</td>
</tr>
<tr>
<td>$v =$ 50 $/unit</td>
</tr>
<tr>
<td>$N =$ 4 locations</td>
</tr>
<tr>
<td>Trk Cap = 500 units/shipment</td>
</tr>
<tr>
<td>DOS = 0.083 years</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

For N Locations

<table>
<thead>
<tr>
<th>Policy</th>
<th>EOQ</th>
<th>FOS</th>
<th>DOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order Size</td>
<td>$q^*$</td>
<td>$q_{DOS}$</td>
<td></td>
</tr>
<tr>
<td>Average IOH</td>
<td>$\sqrt{N(Q^*/2)}$</td>
<td>$N(Q_{FOS}/2)$</td>
<td>$Q_{DOS}/2$</td>
</tr>
<tr>
<td>Order Cost</td>
<td>$\sqrt{N(O_{EOQ})}$</td>
<td>$O_{FOS}$</td>
<td>$N(O_{DOS})$</td>
</tr>
<tr>
<td>Holding Cost</td>
<td>$\sqrt{N(H_{EOQ})}$</td>
<td>$N(H_{FOS})$</td>
<td>$H_{DOS}$</td>
</tr>
<tr>
<td>Total Cost</td>
<td>$\sqrt{N(O_{EOQ}+H_{EOQ})}$</td>
<td>$O_{FOS}+N_{FOS}$</td>
<td>$N_{DOS}+H_{DOS}$</td>
</tr>
</tbody>
</table>

Single Location

<table>
<thead>
<tr>
<th>Policy</th>
<th>EOQ</th>
<th>FOS</th>
<th>DOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order Size</td>
<td>400</td>
<td>500</td>
<td>167</td>
</tr>
<tr>
<td>Average IOH</td>
<td>200</td>
<td>250</td>
<td>83</td>
</tr>
<tr>
<td>Order Cost</td>
<td>$2,500$</td>
<td>$2,000$</td>
<td>$6,000$</td>
</tr>
<tr>
<td>Holding Cost</td>
<td>$2,500$</td>
<td>$3,125$</td>
<td>$1,042$</td>
</tr>
<tr>
<td>Total Cost</td>
<td>$5,000$</td>
<td>$5,125$</td>
<td>$7,042$</td>
</tr>
</tbody>
</table>

4 Locations

<table>
<thead>
<tr>
<th>Policy</th>
<th>EOQ</th>
<th>FOS</th>
<th>DOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order Size</td>
<td>200</td>
<td>500</td>
<td>42</td>
</tr>
<tr>
<td>Average IOH</td>
<td>400</td>
<td>1000</td>
<td>21</td>
</tr>
<tr>
<td>Order Cost</td>
<td>$5,000$</td>
<td>$2,000$</td>
<td>$24,000$</td>
</tr>
<tr>
<td>Holding Cost</td>
<td>$5,000$</td>
<td>$12,500$</td>
<td>$1,042$</td>
</tr>
<tr>
<td>Total Cost</td>
<td>$10,000$</td>
<td>$14,500$</td>
<td>$25,042$</td>
</tr>
</tbody>
</table>
Extensions: Discounts

- **All Units Discount**
  - Discount applies to all units purchased if total amount exceeds the break point quantity
  - Examples?

- **Incremental Discount**
  - Discount applies only to the quantity purchased that exceeds the break point quantity
  - Examples?

- **One Time Only Discount**
  - Less common – but not unheard of!
  - A one time only discount applies to all units you order right now (no quantity minimum or limit)

- How will different discounting strategies impact your lot sizing decision?
- What cost elements are relevant?
Extensions: All Units Discounts

Two Cases to Examine . . .

\[ v = \begin{cases} 
  v_0 & 0 \leq Q \leq Q_b \\
  v_0 (1 - d) & Q_b \leq Q 
\end{cases} \]

\[
TRC = \begin{cases} 
  Dv_0 + \frac{AD}{Q} + \frac{v_0rQ}{2} & 0 \leq Q \leq Q_b \\
  Dv_0 (1 - d) + \frac{AD}{Q} + \frac{v_0 (1 - d) rQ}{2} & Q_b \leq Q 
\end{cases}
\]

Where
- \( d \) = Discount
- \( v_0 \) = Base unit price
- \( Q_b \) = Break quantity

Typically, \( Q^* < Q_b \) but what if \( Q^* > Q_b \)?
Extensions: All Units Discounts

Simple efficient algorithm
1. Find EOQ with discount (EOQ_d)
2. If EOQ_d ≥ Q_b then pick EOQ_d
   Otherwise, go to 3
3. Solve for TRC(Q*) and TRC(Q_b)
   If TRC(Q*) < TRC(Q_b) then pick Q*
   Otherwise, pick Q_b

Can be extended to more than one break point

Example:
D = 2000 Units/yr
r = .25
A = $500
v_0 = $50
Discount of 2% off if Q ≥ 500
Extensions: Incremental Discounts

- Discount only applies to quantity above breakpoint
- Trade-off between lower purchase cost and higher carrying costs
- Cost of units ordered below each breakpoint are essentially ‘fixed’
Extensions: Incremental Discounts

- Efficient algorithm
  1. Find Fixed Cost per breakpoint, $F_i$, for each break
  2. Find $EOQ_i$ for each range – including the $F_i$
  3. If $EOQ_i$ is not within allowable range, go to next $i$
     Otherwise, find TRC$_i$ using effective cost per unit, $v_{ei}$
  4. Pick $EOQ_i$ with lowest TRC

- Can be extended to more than one break point

\[
F_i = F_{i-1} + (v_{i-1} - v_i)Q_i \quad F_0 = 0
\]

\[
EOQ_i = \sqrt{\frac{2D(A + F_i)}{rv_i}}
\]

\[
v_e = \frac{v_iEOQ_i + F_i}{EOQ_i}
\]
**Example: Incremental Discounts**

Price Breaks:
10% off for 500 to <1000 units
20% off for 1000 or more units

<table>
<thead>
<tr>
<th></th>
<th>i=2</th>
<th>i=1</th>
<th>i=0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_i$</td>
<td>$40</td>
<td>$45</td>
<td>$50</td>
</tr>
<tr>
<td>$Q_{bi}$</td>
<td>1,000</td>
<td>500</td>
<td>0</td>
</tr>
<tr>
<td>$F_i$</td>
<td>7,500</td>
<td>2,500</td>
<td>0</td>
</tr>
<tr>
<td>$EOQ_i$</td>
<td>1,789</td>
<td>1,033</td>
<td>400</td>
</tr>
<tr>
<td>$V_{ei}$</td>
<td>$44.19</td>
<td></td>
<td>$50</td>
</tr>
<tr>
<td>Purch Order Hold</td>
<td>$88,384</td>
<td>$100,000</td>
<td>$2,500</td>
</tr>
<tr>
<td>Hold</td>
<td>$9,882</td>
<td>$2,500</td>
<td>$105,000</td>
</tr>
</tbody>
</table>

D=2000 Units/yr  
$r=.25$  
$A=$500  
$v_0 = $50
Suppose you are offered a One Time deal! Should you take it?

\[
\begin{align*}
\nu_g &= \text{One time good deal purchase price ($/unit)} \\
Q_g &= \text{One time good deal order quantity (units)} \\
TC_{sp} &= \text{TC over time covered by special purchase (\$)}
\end{align*}
\]
Extensions: One Time Discount

Compare Options: Not Special Price vs. Special Price

- Find TC for normal price

\[
TC = (CycleTime)(TC^* + PurchaseCost)
\]

\[
TC = \left(\frac{Q_g}{D}\right) \sqrt{2ArvD} + \left(\frac{Q_g}{D}\right)vD
\]

- Find the Savings (TC-TC\text{SP})

\[
Savings = TC - TC_{SP}
\]

\[
= \left(\left(\frac{Q_g}{D}\right) \sqrt{2ArvD} + \left(\frac{Q_g}{D}\right)vD\right) - \left(v_g Q_g + rv_g \left(\frac{Q_g}{2}\right)\left(\frac{Q_g}{D}\right) + A\right)
\]
Extensions: One Time Discount

Finding 1st and 2nd order conditions (Maximize Savings)

\[
\frac{dS}{d(Q_g)} = 0 = \left( \frac{1}{D} \right) \sqrt{2AvrD} + (v - v_g) - \left( \frac{2rv_g Q_g}{2D} \right)
\]

\[
\frac{d^2 S}{d^2 (Q_g)} = -\left( \frac{2rv_g}{2D} \right) < 0
\]

So that the Optimal Quantity to buy is

\[
Q_g^* = \left( \frac{D}{Drv_g} \right) \sqrt{2ArvD} + \frac{D(v - v_g)}{rv_g}
\]

Cleaning this up gives:

\[
Q_g^* = Q^* \left( \frac{v}{v_g} \right) + \frac{D(v - v_g)}{rv_g}
\]
Take-Aways

- EOQ is a good place to start for most analysis
- EOQ can be extended to cover
  - Non-zero leadtimes
  - Finite replenishment systems
  - Multiple locations
    - Square Root law rests on implicit assumptions
    - Distribution of demand and inventory policy will impact results
- Discounts
  - Purchase price (v) becomes relevant
  - Common in practice (economies of scale)
Questions?
Comments?
Suggestions?