Inventory and EOQ Models

Agenda

- Inventory
  - Reasons for holding inventory
  - Dimensions of inventory models
- EOQ-type models
  - Basic model
  - EPQ model
  - Planned backorders
  - Quantity discounts
Inventory

Inventory = cumulative supply – cumulative demand

Why hold inventories?

- The transaction motive
  - Economies of scale: production, transportation, discount, replenishment, ...
  - Competition purpose
- The precautionary motive
  - Demand uncertainty: unpredictable events
  - Supply uncertainty: lead time, random yield, ...
- The speculative motive
  - Fluctuating value: ordering cost, selling price
  - Demand increase: seasonality, promotion, ...

supply → inventory → demand
Dimensions of inventory models

- **Products**
  - single product vs. multiple products
  - perishable or durable
- **Decision variables**
  - when and how much to order
  - pricing
  - production and/or delivery schedule
  - capacity expansion
  - setup reduction
  - quality improvement
- **Decision making structure**
  - single decision maker vs. multiple decision makers
- **Time**
  - single period, finite horizon, infinite horizon
  - deterministic or stochastic

Dimensions of inventory models

- **Objective function**
  - costs (average or discounted): order/production, inventory holding and shortage
  - Profit
  - risk-neutral vs. risk averse
- **Physical system**
  - single location vs. multiple locations
  - single stage vs. multiple stages
- **Information structure**
  - continuous review vs. periodic review
  - inexact stock level
- **Resource constraints**
  - limited capacity
Dimensions of inventory models

- Supply
  - Controllable: when and how much to order
  - Supply contracts
  - Imperfect quality
  - Limited capacity
  - Lead time
- Demand
  - Exogenous: deterministic (constant or time dependent), stochastic
  - Endogenous: pricing model

Ordering costs in inventory models

- Ordering costs
  - Linear: proportional to order quantity
  - Concave: economies of scale, incremental discount
  - General: all-units discount
Inventory costs in inventory models

- **Inventory carrying costs**
  - Insurance cost: 2%
  - Maintenance cost: 6%
  - Opportunity cost of alternative investment: 7-10%

- **Shortage costs**: loss of good will or reputation (hard to quantify)
  - Lost sale case
  - Backorder case

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EOQ Model: Assumptions

- infinite horizon
- constant and deterministic demand: $D$ items/unit time
- no shortages
- fixed order quantity $Q$
- zero lead time
- order cost: $K + cQ$
- inventory holding cost: $h$ per item per unit time

Objective of the EOQ model

Objective: minimize the average cost per unit of time over the infinite horizon subject to no shortages

$$\min \lim_{T \to \infty} \frac{1}{T} \int_0^T \{hI(t) + O(t)\} dt$$

$I(t)$: inventory level at time $t$

$O(t)$: order cost at time $t$
**EOQ Model: Graphical Representation**

**EOQ Model: Costs per unit time**

The total cost for one cycle

\[ = (K + cQ) + h \int_0^{Q/D} I(t) \, dt \]

\[ = (K + cQ) + h \int_0^{Q/D} (Q - Dt) \, dt \]

\[ = (K + cQ) + \frac{hQ^2}{2D} \]

The average cost per unit of time

\[ = \frac{KD}{Q} + \frac{hQ}{2} + cD \]
**EOQ Model: EOQ**

Find the optimal ordering quantity: Economic Order Quantity

First order optimality condition

\[ 0 = \frac{d(Cost \text{ per unit time})}{dQ} = -\frac{KD}{Q^2} + \frac{h}{2} \]

EOQ \(= Q^* = \sqrt{\frac{2DK}{h}} \)

Optimal Cycle Time \(= T^* = \sqrt{\frac{2K}{hD}} \)

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**EOQ Model: Two Questions**

- Lead Time
- Why EOQ independent of the variable ordering cost?
EOQ Model: One Example

A) The demand for electrical components is fixed at a rate of 2400 units/month. Each time the store makes an order it costs 320$. The item costs 3$. The annual inventory holding cost rate is 20%.

\[ Q^* = 5543 \text{ units}, \ T^* = 2.3 \text{ months} \]

EOQ Model: One Example

B) Suppose we now order electrical components only in hundreds of units.

\[ Q^* = 5500 \text{ units} \]
\[ C(5500) = 277.13 + cD \] $
\[ C(5600) = 278 + cD \]$
Sensitivity Analysis

\[
\begin{align*}
C'(Q^*) &= \sqrt{2K Dh} \\
C'(\gamma Q^*) &= \frac{1}{\gamma} \sqrt{\frac{1}{2} K Dh} + \gamma \sqrt{\frac{1}{2} K Dh} \\
&= \sqrt{2K Dh} (\gamma + \frac{1}{\gamma}) / 2 \\
\frac{C(\gamma Q^*)}{C(Q^*)} &= (\gamma + \frac{1}{\gamma}) / 2
\end{align*}
\]

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.5</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
<th>1.2</th>
<th>1.5</th>
<th>2</th>
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<tbody>
<tr>
<td>$\frac{C(\gamma Q^<em>)}{C(Q^</em>)}$</td>
<td>1.25</td>
<td>1.025</td>
<td>1.006</td>
<td>1</td>
<td>1.017</td>
<td>1.083</td>
<td>1.25</td>
</tr>
</tbody>
</table>

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EPQ Model

- Production Planning Model
  - No shortages
  - Production rate: $P > D$

\[ Q - T_p D \]

\[ P > D \quad D \quad T \]

\[ T_p \quad T_D \quad T \]

time

Inventory level

EPQ: Analysis

Let the total demand in one cycle is $Q = T_p P$

The cost for one cycle

\[ K + h \int_0^{T_p} (P - D) t dt + h \int_{T_p}^T (Q - D t) dt \]

\[ = K + \frac{h}{2} (P - D) Q^2 + h \left[ \frac{Q^2}{2D} - \frac{Q^2}{T} + \frac{D Q^2}{T^2} \right] \]

Average cost per unit time $C(Q)$

\[ = \frac{DK}{Q} + \frac{hQ}{2} \frac{P - D}{P} \]

Economic Production Quantity:

\[ \frac{dC(Q)}{dQ} = 0 \rightarrow Q^* = \sqrt{\frac{2DK}{h} \frac{P}{P - D}} \]

\[ C(Q^*) = \sqrt{2DKh} \sqrt{\frac{P - D}{P}} \]
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EOQ Model: Planned Backorder

- Let $\pi$ be the shortage cost per item per unit of time

\[ \text{Inventory level} \]

\[ Q \]

\[ s \]

\[ s-Dt \]

\[ t \]

\[ \frac{s-Dt}{D} \]

\[ \frac{(Q-s)}{D} \]

\[ \text{time} \]
EOQ Model: Backorder Analysis

Total Costs Per Cycle
\[ TC(s, Q) = K + \frac{hs^2}{2D} + \frac{\pi}{2D}(Q - s)^2 \]

Average Cost per unit time
\[ C(s, Q) = \frac{DK}{Q} + \frac{hs^2}{2Q} + \frac{\pi}{2Q}(Q - s)^2 \]

Economic Order Quantity and Order-up-to Level
\[ Q^* = \sqrt{\frac{2KD}{h}} \sqrt{\frac{\pi + h}{\pi}} \]
\[ s^* = \sqrt{\frac{2KD}{h}} \sqrt{\frac{\pi}{\pi + h}} \]
\[ C(s^*, Q^*) = \sqrt{2KDh} \sqrt{\frac{\pi}{\pi + h}} \]

Why shortages?

![Diagram showing EOQ model with backorder analysis](image)
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Quantity Discounts

<table>
<thead>
<tr>
<th>Order cost</th>
<th>Incremental discounts</th>
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<tbody>
<tr>
<td>K</td>
<td>K + c₁q + c₂(Q-q)</td>
</tr>
<tr>
<td>K + c₁Q</td>
<td>K + c₂Q</td>
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<tr>
<td>All-units discounts</td>
<td></td>
</tr>
<tr>
<td>C₁ &gt; C₂</td>
<td></td>
</tr>
</tbody>
</table>
All-units discounts: Case 1

\[ C_1(Q) = \frac{K}{Q} + c_1 D + \frac{hD}{2} \]

\[ C_2(Q) = \frac{K}{Q} + c_2 D + \frac{hD}{2} \]

\[ q^* = \sqrt{\frac{2KD}{h}} \]

All-units discounts: Case 2

\[ C_1(Q) = \frac{K}{Q} + c_1 D + \frac{hD}{2} \]

\[ C_2(Q) = \frac{K}{Q} + c_2 D + \frac{hD}{2} \]

\[ q^* = \sqrt{\frac{2KD}{h}} \]
All-units discounts: Case 3

\[ C_1(Q) = \frac{KD}{Q} + c_1D + hD/2 \]

\[ C_2(Q) = \frac{KD}{Q} + c_2D + hD/2 \]

\[ Q^*_0 = \sqrt{\frac{2KD}{h}} \]

All-units discount: Summary

Case 1: \( q \leq Q^*_0 \), \( Q^* = Q^*_0 \)

Case 2: \( q \geq Q^*_0 \), \( C_1(Q^*_0) \leq C_2(q) \), \( Q^* = Q^*_0 \)

Case 3: \( q \geq Q^*_0 \), \( C_1(Q^*_0) \geq C_2(q) \), \( Q^* = q \)