Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72

CBA 2. The Time Value of Money

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The Importance of Time

• A dollar in hand now is worth more than a dollar received in the future, because of its earning power, i.e., it can be invested to generate income.

• The purchasing power of money, i.e., the amount of goods that a certain amount can buy, changes with time also.

• Objective: To develop methods for establishing the equivalence of sums of money. It depends on the amounts, the time of occurrence of the sums of money, and the interest rate.
Overview of Lecture

• The Basics of Interest Rates – Simple and Compound Interest
• The Basic Discount Factors – Present Value, Future Value, Annual Value
• Economic Equivalence and Net Present Value
• Return to Interest Rates: Nominal and Effective Rates
• Inflation
Simple Interest

- P: Principal amount
- n: Number of interest periods
- i: Interest rate
- I: Interest earned
- Interest and principal become due at the end of n.

\[ I = Pni \]

- The interest is proportional to the length of time the principal amount was borrowed.
Compound Interest

• Interest is payable at the end of each interest period.
• If the interest is not paid, the borrower is charged interest on the total amount owed (principal plus interest).
• Example: $1,000 is borrowed for two years at 6% (compounded). A single payment will be made at the end of the second year.
  • Amount owed at the beginning of year 2: $1,060
  • Amount owed at the end of year 2:
    \[1,060 \times 1.06 = 1,000 \times (1.06)^2 = 1,123.60\]
  • For simple interest, the amount owed at the end of year 2 would be: $1,000 + 1,000 \times 2 \times 0.06 = 1,120.00
Cash Flows over Time

- Up arrow = we receive $; down arrow = we pay $
- Amount borrowed: $1,000
- Interest is paid at the end of each year at the rate of 10%.
- The principal is due at the end of the fourth year.
The Basic Discount Factors

- **P** (present value)
- **F** (future value)
- **A** (annual value or "annuity")
Single-Payment Compound-Amount Factor

- A single payment is made after n periods.
- The interest earned at the end of each period is charged on the total amount owed (principal plus interest).
- $1 now is worth \((F/P, i, n)\) at time n if invested at i%.

\[ (F/P, i, n) = (1+i)^n \]

\[ F = P(1+i)^n \]
Single-Payment Present-Worth Factor

\[(P / F, i, n) = \frac{1}{(1 + i)^n}\]

- The reciprocal of the single-payment compound amount factor.
- Discount rate: \(i\)
- $1 n years in the future is worth \((P/F, i, n)\) now.
Equal-Payment-Series Compound-Amount Factor

- Equal payments, A, occur at the end of each period.
- We will get back \((F/A, i, n)\) at the end of period \(n\) if funds are invested at an interest rate \(i\).
- \(F = A + A(1+i) + A(1+i)^2 + \ldots + A(1+i)^{n-1}\)

\[
(F / A, i, n) = \frac{(1 + i)^n - 1}{i}
\]
Equal-Payment-Series Sinking-Fund Factor

- May be used to determine the payments $A$ required to accumulate a future amount $F$.

$$(A/F, i, n) = \frac{i}{(1+i)^n - 1}$$

**Example.** We wish to deposit an amount $A$ every 6 months for 3 years so that we’ll have $10,000 at the end of this period. The interest rate is 5% per year.

- $n = 6$ deposits
- $i = 2.5\%$ per 6-month period
- $F = \$10,000$
- $(A/F, 0.025, 6) = 0.15655 \Rightarrow A = \$1,565.50$
Equal-Payment-Series Capital-Recovery Factor (1)

• An amount P is deposited now at an annual interest rate i.

• We will withdraw the principal plus the interest in a series of equal annual amounts A over the next n years.

• The principal will be worth P(1+i)^n (slide 8) at the end of n years. This amount is to be recovered by receiving A every year ⇒ the sinking-factor formula applies (slide 11) ⇒
Equal-Payment-Series Capital-Recovery Factor (2)

\[ A = P(1+i)^n \left[ \frac{i}{(1+i)^n - 1} \right] \quad \Rightarrow \quad (A/P, i, n) = \frac{i(1+i)^n}{(1+i)^n - 1} \]

**Example:** Your house mortgage is $300,000 for 30 years with an nominal annual rate of 7%. What is the monthly payment?

\[ n = 360 \text{ months} \quad \text{and} \quad i = 0.583\% \text{ per month} \]

\[ (A/P, 0.00583, 360) = 0.006650339 \Rightarrow A = 300,000 \times 0.006650339 = $1,995.10 \text{ per month} \]
Summary of the Formulas

Single-Payment Compound-Amount Factor

\[(F/P, i, n) = (1+i)^n\]

Equal-Payment-Series Compound-Amount Factor

\[(F/A, i, n) = \frac{(1+i)^n - 1}{i}\]

Equal-Payment-Series Capital-Recovery Factor

\[(A/P, i, n) = \frac{i(1+i)^n}{(1+i)^n - 1}\]
Continuous Compounding (1)

- Suppose that interest is compounded a very large number of times. Then, the effective annual interest rate is

\[ i = \lim_{m \to \infty} \left( 1 + \frac{r}{m} \right)^m - 1 = e^r - 1 \]

where \( r \) is the nominal annual interest rate.
Continuous Compounding (2)

\[(F/P, i, n) = (1+i)^n\]

\[(F/P, r, n) = e^{rn}\]

\[(F/A, i, n) = \frac{(1+i)^n - 1}{i}\]

\[(F/A, r, n) = \frac{e^{rn} - 1}{e^r - 1}\]

\[(A/P, i, n) = \frac{i(1+i)^n}{(1+i)^n - 1}\]

\[(A/P, r, n) = \frac{e^r - 1}{1 - e^{-rn}}\]
Nominal and Effective Interest Rates (1)

- **The nominal interest rate (or annual percentage rate)** is the annual rate without the effect of any compounding.

- **The effective (actual) interest rate** is the annual rate taking into account the effect of any compounding during the year.

**Example:** A credit card advertises a nominal rate of 18% compounded monthly. The actual rate is, then, \((18/12) = 1.5\%\) per month. The effective annual rate is \((1.015)^{12} - 1 = 0.1956\) or 19.56\% (if you do not pay anything each month)
The effective interest rate $i$ depends on the frequency of compounding.

Example: nominal interest rate $r = 10\%$
- **Compounded annually**: $i = r = 10\%$
- **Compounded quarterly**: $i = (1+0.1/4)^4 -1 = 10.38\%$
- **Compounded monthly**: $i = (1+0.1/12)^{12} -1 = 10.471\%$
- **Compounded weekly**: $i = (1+0.1/52)^{52} -1 = 10.506\%$
- **Compounded daily**: $i = (1+0.1/365)^{365} -1 = 10.516\%$
- **Compounded continuously**: $i = e^{0.1} - 1 = 10.517\%$
Nominal and Effective Interest Rates (3)

In the formulas we introduced in earlier slides, $i$ is the effective interest rate for a given period and $n$ is the number of such periods.

**Example:** You wish to buy a house and you can afford to make a down payment of $50,000. Your monthly mortgage payment cannot exceed $2,000. If 30-year loans are available at 7.5% interest compounded monthly, what is the highest price that you may consider?
Nominal and Effective Interest Rates (4)

Solution:

Let’s use *one month* as the time period. Then, $n = 360$ months, and $i = (7.5/12) = 0.625\%$. We know that

$$\frac{i(1+i)^n}{(1+i)^n - 1} = \frac{\text{A/P}, 0.00625, 360}{0.00699} = \frac{\text{P}}{0.00699} = \frac{\text{H} - 50,000}{0.00699} \leq 2,000 \Rightarrow \text{H} \leq \frac{2,000}{0.00699} + 50,000 = $336,123\]
Nominal and Effective Interest Rates (5)

- Let’s use one year as the time period. Then, n = 30 years, and $i = (1 + 0.00625)^{12} - 1 = 7.763\%$

Then, $(A^*/P, 0.07763, 30) = 0.0867 \quad A^* = 0.0867P$ per year

Your effective payment per year is $A^* = 2000 \times (F/A, 0.00625, 12) = 2000 \times 12.4212 = 24,842$

$P \leq (24,842/0.0867) + 50,000 = 336,533 \quad$ as before

Consistency between $i$ and $n$ will lead to identical solutions.
Economic Equivalence (1)

• The formulas that we have developed establish economic equivalence between $P$ and $F$, an equal-payment series and $F$, and so on.

*Example:* Consider the following cash flow: You will receive $500 at the end of years 3 and 4 and $1,000 at the end of year 5. If the interest rate is 7%, what amount received at the present is equivalent to this cash flow?
Economic Equivalence (2)

Solution:

\[ P = \frac{500}{(1 + 0.07)^3} + \frac{500}{(1 + 0.07)^4} + \frac{1,000}{(1 + 0.07)^5} \]

\[ P = 408.15 + 381.45 + 712.99 = $1,502.59 \]
Economic Equivalence (3)

If the interest is compounded continuously, the result will be:

\[ P = \frac{500}{e^{3 \times 0.07}} + \frac{500}{e^{4 \times 0.07}} + \frac{1000}{e^{5 \times 0.07}} \]

\[ P = 405.30 + 377.89 + 704.69 = $1,487.88 \]
Inflation

- The *purchasing power* of money declines when the prices increase.
- This must be included in equivalence calculations.
- A *price index* is the ratio of the price of a commodity or service at some point in time to the price at some earlier point.
- The *Consumer Price Index (CPI)* represents the change in prices of a “market basket,” that includes clothing, food, utilities, and transportation.
- The CPI measures the changes in retail prices to maintain a fixed standard of living for the “average” consumer.
# CPI and Inflation

<table>
<thead>
<tr>
<th>Year</th>
<th>Consumer Price Index (CPI)</th>
<th>(Annual Rate of Inflation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1967</td>
<td>100.0</td>
<td>2.9%</td>
</tr>
<tr>
<td>1972</td>
<td>125.3</td>
<td>3.3%</td>
</tr>
<tr>
<td>1977</td>
<td>181.5</td>
<td>6.5%</td>
</tr>
<tr>
<td>1980</td>
<td>246.8</td>
<td>13.5%</td>
</tr>
<tr>
<td>1985</td>
<td>322.2</td>
<td>3.6%</td>
</tr>
<tr>
<td>1990</td>
<td>391.4</td>
<td>5.4%</td>
</tr>
<tr>
<td>1995</td>
<td>456.5</td>
<td>2.8%</td>
</tr>
<tr>
<td>1999</td>
<td>497.6</td>
<td>1.9%</td>
</tr>
</tbody>
</table>

Inflation Rate

Annual inflation rate for year $t+1$:

$$f_{t+1} = \frac{CPI_{t+1} - CPI_t}{CPI_t}$$

For many calculations, an average inflation rate is sufficient.

$$CPI_t(1+f)^n = CPI_{t+n}$$

Note: Thuesen & Fabrycky use $\bar{f}$ for the average rate.
Example

The average inflation rate from 1967 to 1999 is given by

\[ 100(1+f)^{32} = 497.6 \quad \Rightarrow \quad 1+f = 4.976^{1/32} = 1.0514 \]

\[ \Rightarrow \quad f = 5.14\% \]
Definitions

• *Market interest rate (or current-dollar interest rate) i*: The interest rate available in finance. Inflation impact is included.

• *Inflation-free interest rate (or constant-dollar interest rate) i'*: It represents the earning power of money with inflation removed. It must be calculated.

• *Actual dollars*: The amount received or disbursed at any point in time.

• *Constant dollars*: The hypothetical amount received or disbursed in terms of the purchasing power of dollars at some base year.
Constant and Actual Dollars

• (actual dollars) = (1+f)\(^n\) (constant dollars)  
(based on the purchasing power n years earlier)

• Equivalence in terms of actual dollars: Use i.

• Equivalence in terms of constant dollars: Use i'.

• Relationship among i, i', and f:

\[
i' = \frac{1+i}{1+f} - 1
\]
Proof

Proof: At the base year (t=0), constant and actual dollars coincide.
Let P be the present value. Then, n years from now,

\[ F = (1+i)^n P \quad \text{actual dollars} \]
\[ F' = (1+ i')^n P \quad \text{constant dollars} \quad \Rightarrow \]
\[ F = (1+f)^n F' = (1+f)^n (1+ i')^n P \quad \text{actual dollars} \quad \Rightarrow \]

\[ i' = \frac{1+i}{1+f} - 1 \]
Example: Going to the Movies

- **1967 Ticket Price:** $1.25
- **1999 Ticket Price:** $8.50
- **Has there been a price increase above the rate of inflation?**

The average rate of inflation has been (slide 28): $f = 5.14\%$. The actual rate of increase is $i = (8.5/1.25)^{1/32} - 1 = 0.0617$. Therefore, $i' = [(1+0.0617)/(1+0.0514)] - 1 = 0.0098 \approx 1\%$
Example: Investments in Two Countries (1)

John has immigrated to the US where the inflation rate is 2% while his brother Joe has stayed in the old country where the inflation rate is 4.5%. The US banks give an interest rate of 5.5% while those of the old country give 8%.

1. What are the real interest rates in the two countries?

\[ i'_{US} = \frac{1 + 0.055}{1 + 0.02} - 1 = 3.43\% \]
\[ i'_{OC} = \frac{1 + 0.08}{1 + 0.045} - 1 = 3.35\% \]
Example: Investments in Two Countries (2)

2. If John decides to invest in the Old Country, what would his real interest rate be?

\[ i'_{US/OC} = \frac{1 + 0.08}{1 + 0.02} - 1 = 5.88\% \]

Interest rate of the OC (John invests there)

US inflation rate (John lives there)