Broken stick experiment

\[ D = \text{Min}[X_1, X_2] \]

\[ D = \text{Max}[X_1, X_2] \]

\[ S = X_1 + X_2 \]

Convolution

Functions of Random Variables
But first, we have a winner!

- The winning submission for ESD.86, *for most blatant misuse, abuse or misinterpretation of statistics and probability in the media.*
- Submitted by Roberto Perez-Franco.

- Original article *New York Times:*
  
  51% of Women Are Now Living Without Spouse, *New York Times*, January 16, 2007, Section A; Column 1; National Desk; Pg. 1

**Today:** HEARING ON 'WARMING OF PLANET’ CANCELED BECAUSE OF ICE STORM
Problem Framing, Formulation and Solution
Break a yardstick in two random places

What is the probability that a triangle can be formed with the resulting three stick pieces?
Breaking a Stick

- Mark the stick....
Marking the Results

$X_2$  $X_1$
1. Random Variables:

\[ X_1 = \text{location of first mark} \]
\[ X_2 = \text{location of second mark} \]
Step 2: Joint Sample Space
Step 3: Probability Uniform over the Square
Step 4: Carefully Work Within the Sample Space

- What conditions need to be satisfied so that a triangle can be formed?
- Suppose we consider first the case shown, $x_1 > x_2$
After Step 4, **HAPPINESS!**

http://web.mit.edu/urban_or_book/www/animated-eg/stick/f1.0.html
Functions of Random Variables

\[ Y = 3X - 2Z \]
4 Steps:

1. Define the Random Variables
2. Identify the joint sample space
3. Determine the probability law over the sample space
4. Carefully work in the sample space to answer any question of interest
4 Steps: Functions of R.V.s

1. Define the Random Variables
2. Identify the joint sample space
3. Determine the probability law over the sample space
4. Carefully work in the sample space to answer any question of interest

4a. Derive the CDF of the R.V. of interest, working in the original sample space whose probability law you know
4b. Take the derivative to obtain the desired PDF
Photos of ambulance and a dispatch center removed due to copyright restrictions.
Response Distance of an Ambulance

1. R.V. s
   - $X_1$ = location of the accident
   - $X_2$ = location of the ambulance
   - $D = \text{response distance} = |X_1 - X_2|$

2. Joint sample space is unit square in $X_1$ $X_2$ plane

3. PDF over square is uniform
$$x_2 = x_1 + y$$

$$x_2 = x_1 - y$$

$$x_2 > x_1$$

$$x_2 < x_1$$

Event \( \{D < y\} \)
4.a  $F_D(y) = P\{D<y\} = 1 - (1-y)^2$, $0<y<1$
4.b  $f_D(y) = 2(1-y)$, $0<y<1$. 
$$f_D(y) = 2(1-y)$$
In previous problem, $E[D] = 1/3$
What if we fix the location of the ambulance at $X_2 = 1/2$?
fixed location ambulance

E[D] = 1/4, a 25% reduction
Rectangular Response Area

\[ D = |X_1 - X_2| + |Y_1 - Y_2| \]
Scaling to Get Expected Travel Distance

\[ D = |X_1 - X_2| + |Y_1 - Y_2| \]

\[ E[D] = E[|X_1 - X_2| + |Y_1 - Y_2|] \]

\[ E[D] = (1/3)[X_0 + Y_0] \]
More Examples of Functions of Random Variables
1. Define the Random Variables

\[ Y = \text{MIN}\{X_1, X_2\}, \text{ where } X_1 \text{ and } X_2 \text{ are iid uniform over } [0, 1] \]

- Identify the joint sample space

3. Determine the probability law over the sample space - uniform
4. Carefully work in the sample space to answer any question of interest.

4a. Derive the CDF of the R.V. of interest, working in the original sample space whose probability law you know.

4b. Take the derivative to obtain the desired PDF.

\[ F_Y(y) = P\{Y < y\} = 1 - (1 - y)^2 \]

\[ f_Y(y) = \frac{d}{dy}[F_Y(y)] \]

\[ f_Y(y) = 2(1-y), \quad 0 < y < 1 \]
Now suppose
$Y = \text{MIN}\{X_1, X_2, X_3, \ldots X_N\}$, where $X_i$ are iid uniform over $[0,1]$

$F_Y(y) = P\{Y < y\} = 1 - P\{Y > y\}$

$F_Y(y) = 1 - (1-y)^N$

$f_Y(y) = \frac{d}{dy} F_Y(y) = N(1-y)^{N-1}; N=1,2,\ldots$

$0 < y < 1$
Now suppose

$$Y = \text{MAX}\{X_1, X_2, X_3, \ldots X_N\},$$
where $X_i$ are iid uniform over $[0,1]$

$$F_Y(y) = P\{Y < y\} = y^N \quad \text{Why?}$$

$$f_Y(y) = Ny^{N-1} \quad N=1,2,\ldots; \quad 0<y<1$$

OK, so now we can do Max and Min.
Sums of Random Variables
Now let
\( Y = X_1 + X_2 \), where \( X_1 \) and \( X_2 \) are iid uniform over \([0,1]\)

\[
F_Y(y) = P\{Y \leq y\} = \begin{cases} 
  y^2/2 & 0 \leq y \leq 1 \\
  1 - (2 - y)^2/2 & 1 \leq y \leq 2 
\end{cases}
\]

\[
f_Y(y) = \begin{cases} 
  y & 0 \leq y \leq 1 \\
  2 - y & 1 \leq y \leq 2 
\end{cases}
\]

\[
f_Y(y) dy = \int_{v=0}^{v=1} f_{X_1}(v) f_{X_2}(y-v) dv dy
\]

Convolution
\[ f_Y(y) = \begin{cases} 
 y & 0 \leq y \leq 1 \\
 1-y & 1 \leq y \leq 2 
\end{cases} \]

\[ f_Y(y) \, dy = \int_{v=0}^{v=1} f_{x1}(v) f_{x2}(y-v) \, dv \, dy \]

**Convolution**

\[ f_{x2}(y-v) \]

\[ f_{x1}(v) \]
\[ f_Y(y) = \begin{cases} 
  y & 0 \leq y \leq 1 \\
  1-y & 1 \leq y \leq 2 
\end{cases} \]

\[ f_Y(y)dy = \int_{v=0}^{v=1} f_{x_1}(v)f_{x_2}(y-v)dvdy \]
$$f_Y(y) = \begin{cases} y & 0 \leq y \leq 1 \\ 1-y & 1 \leq y \leq 2 \end{cases}$$

$$f_Y(y)dy = \int_{v=0}^{v=1} f_{x1}(v)f_{x2}(y-v)dvdy$$

Convolution
A Quantization Problem
Barges in Action

Photo courtesy of Eddie Codel.
http://www.flickr.com/photos/ekai/15899569/
Marine Transfer Station

Courtesy of Dattner Architects. Used with permission.

NYC Marine Transfer Station

Tug Delivers LIGHTS

Tug Picks Up HEAVIES

LIGHT and HEAVY Barges Stored

Loading Barge

Barges Shifted By Hand Or Tug

Refuse Inflow $\lambda_i(t)$
1. The R.V.’s

- $D =$ barge loads of garbage produced on a random day (continuous r.v.)
- $\Theta =$ fraction of barge that is filled at beginning of day ($0 < \Theta < 1$)
- $K =$ total number of completely filled barges produced by a facility on a random day ($K$ integer)
- $K = \lfloor D + \Theta \rfloor =$ integer part of $D + \Theta$
2. The Sample Space
$\theta$

$K = 0$

$K = 1$

$K = 2$

$K = 3$

$d$
3. Joint Probability Distribution
   a) $D$ and $\Theta$ are independent.
   b) $\Theta$ is uniformly distributed over $[0, 1]$
3. Joint Probability Distribution
a) $D$ and $\Theta$ are independent.
b) $\Theta$ is uniformly distributed over $[0, 1]$

$$f_{D, \Theta}(d, \theta) = f_D(d) f_{\Theta}(\theta) = f_D(d)(1) = f_D(d), \ d > 0, \ 0 < \theta < 1$$
4. Working in the Joint Sample Space

Look at $E [K \mid D = d ]$

Let $d = i + x \quad 0 < x < 1$

$E [K \mid D = i + x ] = i (1 - x) + (i + 1) x = i + x = d$

*Implies* $E [K] = E [D]$

Data Collection Implications? Quantized Data?
What Have We Learned Today?

4 Steps: Functions of R.V.s

1. Define the Random Variables
2. Identify the joint sample space
3. Determine the probability law over the sample space
4. Carefully work in the sample space to answer any question of interest
   
4a. Derive the CDF of the R.V. of interest, working in the original sample space whose probability law you know
4b. Take the derivative to obtain the desired PDF