ESD: Recitation #4
Birthday problem
An approximate method

• Bernoulli trials
• Number of trials to compare birthdays of all people in the class:

\[ N = \frac{n!}{(n-2)!2!} = \frac{n!}{2(n-2)!} \]

• Probability that nobody has the same birthday than someone else:

\[ P_0 = \binom{N}{0} \left( \frac{1}{365} \right)^0 \left( \frac{364}{365} \right)^N = \left( \frac{364}{365} \right)^N = \left( \frac{364}{365} \right)^{\frac{n!}{2(n-2)!}} \]
The exact solution

• Probability that nobody has the same birthday than anybody else:

\[ P_0 = \prod_{i=0}^{n-1} \left( 1 - \frac{i}{365} \right) = \frac{365!}{365^n (365 - n)!} \]
What was the average travel distance between two random points in Budapest in the 1850s?
Budapest = Buda + Pest

Photo removed due to copyright restrictions.
The Danube River through Budapest, showing the two shores.
Only one bridge: Széchenyi Lánchid (Chain bridge)
Modeling

Pest: $\lambda_p$

Buda: $\lambda_B$

$w_p$

$w_B$

$I_p$

$I_B$

202 m
Within each city

• In Buda:

\[
P_{B-B} = \left( \frac{w_B \cdot l_B \cdot \lambda_B}{w_B \cdot l_B \cdot \lambda_B + w_P \cdot l_P \cdot \lambda_P} \right)^2
\]

\[
D_B = \frac{1}{3} (w_B + l_B)
\]

• In Pest:

\[
P_{P-P} = \left( \frac{w_P \cdot l_P \cdot \lambda_P}{w_B \cdot l_B \cdot \lambda_B + w_P \cdot l_P \cdot \lambda_P} \right)^2
\]

\[
D_P = \frac{1}{3} (w_P + l_P)
\]
Between the two cities

• 4 cases:
Between (1) and (3)

- **Probability:**

\[ P_{(1)-(3)} = 2 \frac{w_P \cdot l_P \cdot \lambda_P \cdot w_B \cdot l_B \cdot \lambda_B}{(w_B \cdot l_B \cdot \lambda_B + w_P \cdot l_P \cdot \lambda_P)^2} \times \frac{u \cdot v}{l_P \cdot l_B} \]

- **Average Distance:**

\[ \overline{D_{(1)-(3)}} = \frac{1}{2} (w_B + v) + \frac{1}{2} (w_P + u) + 202 \]
Between (1) and (4)

- **Probability:**
  
  \[
  P_{(1)-(4)} = 2 \frac{w_p \cdot l_p \cdot \lambda_p \cdot w_B \cdot l_B \cdot \lambda_B}{(w_B \cdot l_B \cdot \lambda_B + w_p \cdot l_p \cdot \lambda_p)^2} \times \frac{u \cdot (l_B - \nu)}{l_p \cdot l_B}
  \]

- **Average Distance:**
  
  \[
  D_{(1)-(4)} = \frac{1}{2} (w_p + u) + \frac{1}{2} (w_B + (l_B - \nu)) + 202
  \]
And continue…

• Between (2) and (3)
• Between (2) and (4)

• Get the final answer…
More complications

• There is currently ten bridges on the Danube.

• How does average traveling distance change if we build another one?
Bertrand’s Paradox

Joseph Louis François Bertrand
(1822-1900)

Wrote *Calcul des probabilités* in 1888.
The question

• Consider an equilateral triangle inscribed in a circle. Suppose a cord of the circle is chosen at random.

• What is the probability that the chord is longer than a side of the triangle?
Random endpoints

Figure by MIT OCW.
Random radius

Figure by MIT OCW.
Random midpoints

Figure by MIT OCW.
Barbershop

- One barber, two chairs for waiting customers.
- Prospective customers arrive in a Poisson manner at the rate of $\lambda$ per hour.
- It takes the barber $1/\mu$ on average to serve a customer.
- Prospective customers finding the barbershop full are lost forever.
- What is the average number of customers?
Model

\[ \lambda = 2 \mu \]
Solving (1)

• What is the probability that \( N \) customers are in the barbershop?

\[
P_1 = \frac{\lambda}{\mu} P_0; P_2 = \frac{\lambda}{\mu} P_1
\]

\[
P_N = \left( \frac{\lambda}{\mu} \right)^N P_0
\]

\[
\sum_{i=0}^{3} P_i = 1 \Rightarrow P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left( \frac{\lambda}{\mu} \right)^{3+1}}
\]
Solving (2)

- Average number of customers:

\[ P_N = \left( \frac{\lambda}{\mu} \right)^3 \frac{1 - \frac{\lambda}{\mu}}{\left(1 - \left( \frac{\lambda}{\mu} \right) \right)^{3+1}} = \frac{2^3}{2^4 - 1} \]

\[ E[Nb\_customers] = \sum_{i=0}^{3} i.P_i = \frac{34}{15} \approx 2.2667 \]