ESD: Recitation #5
The barbershop revisited

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Infinite number of waiting seats

• One barber, infinite number of chairs for waiting customers.
• Prospective customers arrive in a Poisson manner at the rate of $\lambda$ per hour.
• It takes the barber $1/\mu$ on average to serve a customer ($\lambda = 0.9 \times \mu$).
• No prospective customer is ever lost.
• What is the average number of customers?
Model

\[
\begin{align*}
\lambda & \quad \lambda & \quad \lambda & \quad \lambda \\
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & \ldots \\
\mu & \quad \mu & \quad \mu & \quad \mu & \quad \mu
\end{align*}
\]
Solving (1)

• What is the probability that $N$ customers are in the barbershop?

\[ P_1 = \frac{\lambda}{\mu} P_0; P_2 = \frac{\lambda}{\mu} P_1 \]

\[ P_N = \left(\frac{\lambda}{\mu}\right)^N P_0 \]

\[ \sum_{i=0}^{\infty} P_i = 1 \Rightarrow P_0 = \frac{\lambda}{\mu} = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k} = \frac{1}{\sum_{k=0}^{\infty} 0.9^k} = \frac{1}{10} = 0.1 \]
Solving (2)

- Average number of customers:

\[ P_N = 0.1 \times \left( \frac{\lambda}{\mu} \right)^N = 0.1 \times 0.9^N \]

\[ E[Nb\_customers] = \sum_{k=0}^{\infty} k.P_k = 0.1 \times \sum_{k=0}^{\infty} k \times 0.9^k = 9 \]
Different service completion rate

- One barber, two chairs for waiting customers.
- Prospective customers arrive in a Poisson manner at the rate of $\lambda$ per hour.
- It takes the barber $1/\mu$ on average to serve a customer. The service completion rate is described by a second order Erlang pdf. Assume $\lambda = \mu$.
- Prospective customers finding the barbershop full are lost forever.
- What is the average number of customers?
Model

Arrival rate: $f_A(x) = \lambda \cdot e^{-\lambda \cdot x}, \ x \geq 0$
Service rate: $f_S(x) = 4 \cdot \mu^2 \cdot x \cdot e^{-2 \cdot \mu \cdot x}, \ x \geq 0$
Solving (1)

• Steady-state probabilities:

\[ \lambda P_0 = 2\mu P_1 \]
\[ (2\mu + \lambda) P_1 = 2\mu P_2 \]
\[ (2\mu + \lambda) P_1 = 2\mu P_2 \]
\[ (2\mu + \lambda) P_2 = \lambda P_0 + 2\mu P_3 \]
\[ (2\mu + \lambda) P_3 = \lambda P_1 + 2\mu P_4 \]
\[ (2\mu + \lambda) P_4 = \lambda P_2 + 2\mu P_5 \]
\[ 2\mu P_5 = \lambda P_3 + 2\mu P_6 \]
\[ 2\mu P_6 = \lambda P_4 \]
Solving (2)

- Calculate $P_0$:

\[
\sum_{k=0}^{6} P_k = 1 \iff P_0 + \frac{1}{2} P_0 + \frac{3}{4} P_0 + \frac{5}{8} P_0 + \frac{11}{16} P_0 + \frac{21}{32} P_0 + \frac{11}{32} P_0 = \frac{65}{16} P_0
\]

\[
\sum_{k=0}^{6} P_k = 1 \iff P_0 = \frac{16}{65}
\]
Solving (3)

- Average number of customers:

\[
E[Nb\_customers] = 0 \times \frac{16}{65} + 1 \times \left( \frac{8}{65} + \frac{12}{65} \right) + 2 \times \left( \frac{2}{13} + \frac{11}{65} \right) + 3 \times \left( \frac{21}{130} + \frac{11}{130} \right) \\
E[Nb\_customers] = \frac{22}{13} \approx 1.692
\]
Additional barber

- Two barbers:
  - Adam (takes $1/\mu_1$ on average to serve a customer)
  - Ben (takes $1/\mu_2$ on average to serve a customer)
- One chair for waiting customers.
- Prospective customers arrive in a Poisson manner at the rate of $\lambda$ per hour.
- Prospective customers finding the barbershop full are lost forever
Modeling the system

State of the system: $S_{i,j}$

- $i$: number of people being serviced by or waiting for Adam
- $j$: number of people being serviced by Ben
The question

• Suppose $\lambda = \mu_1 = \mu_2$.
• What is the probability that Ben is busy at a random time?
Solving (1)

• Steady-state probabilities:

\[
\begin{align*}
\lambda P_{0,0} &= \mu_1 P_{1,0} + \mu_2 P_{0,1} \\
(\mu_1 + \lambda) P_{1,0} &= \lambda P_{0,0} + \mu_1 P_{2,0} + \mu_2 P_{1,1} \\
(\mu_1 + \lambda) P_{2,0} &= \lambda P_{1,0} + \mu_2 P_{2,1} \\
(\mu_1 + \mu_2) P_{2,1} &= \lambda P_{2,0} + \lambda P_{1,1} \\
(\mu_1 + \mu_2 + \lambda) P_{1,1} &= \lambda P_{0,1} + \mu_1 P_{2,1} \\
(\mu_2 + \lambda) P_{0,1} &= \mu_1 P_{1,1}
\end{align*}
\]
Solving (2)

• \( P\{\text{Ben busy}\} = P_{0,1} + P_{1,1} + P_{2,1} \)

• Using:

\[
P_{0,0} + P_{1,0} + P_{2,0} + P_{2,1} + P_{1,1} + P_{0,1} = 1
\]

• We find:

\[
P\{\text{Ben _ busy}\} = \frac{8}{129} + \frac{4}{43} + \frac{22}{129} = \frac{42}{129} \approx 0.326
\]