ES.1803:   Complex Impedance and Phasors

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1   Introduction

We are going to use the exponential response formula and complex arithmetic to understand the notions of impedance and phasor diagrams for electrical circuits.

We'll see that capacitors and inductors in a circuit have an impedance, which generalizes the notion of resistance across a resistor. (In fact, we will consider resistance to be a type of impedance.) We will see that the rules for combining impedances in series or parallel are exactly the same as those for resistance.

The phasor diagram tells us about the output current and the voltages across the various components in a circuit. It captures graphically the phase angle and phase differences between them.

2   Complex gain

If we have a linear, time invariant system with input $B\cos(\omega t)$ then we know the output is $g(\omega)B\cos(\omega t - \phi(\omega))$. Here, $g(\omega)$ is the real gain and $\phi(\omega)$ is the phase lag.

If we use complex replacement to change the input $B\cos(\omega t)$ to $Be^{i\omega t}$, then the output is $G(i\omega)Be^{i\omega t}$. We call $G(i\omega)$ the complex gain.

Example 1. Consider the system $X'' + 8X' + 7X = F'(t)$, with $F(t)$ the input. Find the real gain, phase lag and complex gain.

(Note: For circuits, we will use capital letters for variables and parameters. We will also add tildes to indicate a complexified sinusoidal function.)

Solution: To find gain and phase lag, we let the input be $F(t) = B\cos(\omega t)$. We start by complexifying the equation:

$$\tilde{X}'' + 8\tilde{X}' + 7\tilde{X} = (Be^{i\omega t})' = i\omega Be^{i\omega t}, \quad \text{where } X = \text{Re}(\tilde{X}).$$

We then use the exponential response formula to solve this. The characteristic polynomial is $P(r) = r^2 + 8r + 7$. So the ERF implies

$$\tilde{X}_p(t) = \frac{i\omega Be^{i\omega t}}{P(i\omega)} = \frac{i\omega}{7 - \omega^2 + 8i\omega} \cdot Be^{i\omega t}.$$

This gives us the complex gain, $G(i\omega) = \dfrac{i\omega}{7 - \omega^2 + 8i\omega}$.

Now, we write $G(i\omega)$ in polar form, $\quad G(i\omega) = |G(i\omega)|e^{-i\phi(\omega)}$.

$|G(i\omega)| = \dfrac{\omega}{\sqrt{(7 - \omega^2)^2 + 64\omega^2}}$. Since $\text{Arg}(7 - \omega^2 + 8i\omega) = \tan^{-1}\left(\dfrac{8\omega}{7 - \omega^2}\right)$ in Q1 or Q2, we have

$$\phi(i\omega) = -\text{Arg}(G(i\omega)) = -(\text{Arg}(i\omega) - \text{Arg}(7 - \omega^2 + 8i\omega)) = -\frac{\pi}{2} + \tan^{-1}\left(\frac{8\omega}{7 - \omega^2}\right) \quad \text{in Q1 or Q4.}$$

So,

$$\tilde{X}_p(t) = \frac{w}{\sqrt{(7-\omega^2)^2 + 64\omega^2}} \cdot Be^{i(\omega t - \phi(\omega))} \quad \Rightarrow \quad X_p(t) = \text{Re}(\tilde{X}_p) = \frac{w}{\sqrt{(7-\omega^2)^2 + 64\omega^2}} B\cos(\omega t - \phi(\omega)).$$

Thus, the gain is $g(\omega) = \dfrac{w}{\sqrt{(7-\omega^2)^2 + 64\omega^2}}$ and the phase lag is $\phi(\omega)$.

Side notes: 1. $G(i\omega) = g(\omega)e^{-\phi(w)}$.

2. If we let the input be e^{st}, where s is any complex number, then the output is $G(s)e^{st}$. $G(s)$ is known as the transfer or system function of the system.

3   General definition of impedance

Later we will discuss the meaning of impedance in circuits. For now, we can give a definition in terms of systems.

For a linear, time invariant system, we assume the input is $B\cos(\omega t)$. For this input, the complex impedance is $Z(\omega) = \dfrac{1}{G(i\omega)}$, where $G(i\omega)$ is the complex gain.

In Example 1, the complex impedance is $Z(\omega) = \dfrac{1}{G(i\omega)} = 7 - \omega^2 + 8i\omega$.

The real impedance is $|Z(\omega)| = \dfrac{1}{|G(i\omega)|} = \dfrac{1}{g(\omega)}$.

3.1   Resistance as impedance

Consider a circuit that consists of a, possibly time varying, input voltage $V_{in}(t)$ in volts and a resistor with resistance R in Ohms. The voltage drives a current $I(t)$ around the circuit.
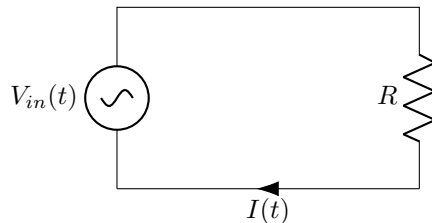
Ohm's law says $V_{in}(t) = RI(t)$ or $I(t) = \dfrac{1}{R} \cdot V_{in}(t)$.

We can consider this to be a system with input $V_{in}(t)$ and output $I(t)$. (It is a very simple linear, time invariant system.)

If we have complexified input $\tilde{V}_{in}(t) = e^{i\omega t}$, then the complexified system is $R\tilde{I} = \tilde{V}_{in}(t)$. So, $\tilde{I}(t) = \frac{1}{R}\tilde{V}_{in}(t)$. This implies the complex gain is $G(i\omega) = \dfrac{1}{R}$

Thus, the impedance $Z(\omega) = \dfrac{1}{G(i\omega)} = R$. For this simple circuit, the impedance is the same as the resistance.
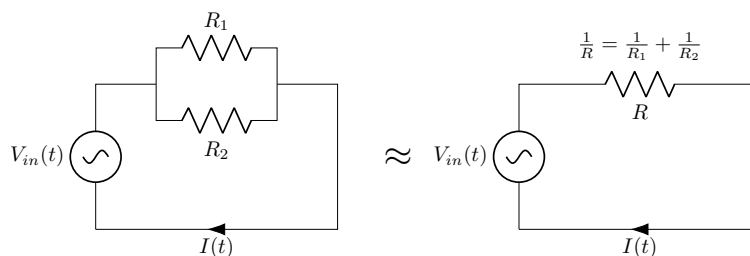
3.2 Resistance in series and parallel

For future reference, let's recall that two circuits with two resistors R_1 and R_2 can be reduced to an equivalent circuit with a single resistor R.

If R_1 and R_2 are in series, then $R = R_1 + R_2$.
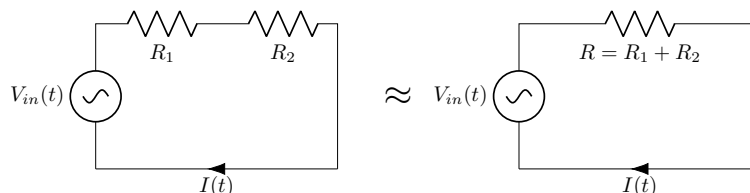
Resistors in series: Both circuits produce the same output $I(t)$ for a given input.

If R_1 and R_2 are in parallel, then R satisfies the formula $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.

Resistors in parallel: Both circuits produce the same output $I(t)$ for a given input.

Summary: For resistors in series, add the resistances; For resistors in parallel, add the complex gains. The same will be true of impedances in general.

4   Simple circuit physics

The picture at right shows an inductor, capacitor and resistor in series with a driving voltage source.
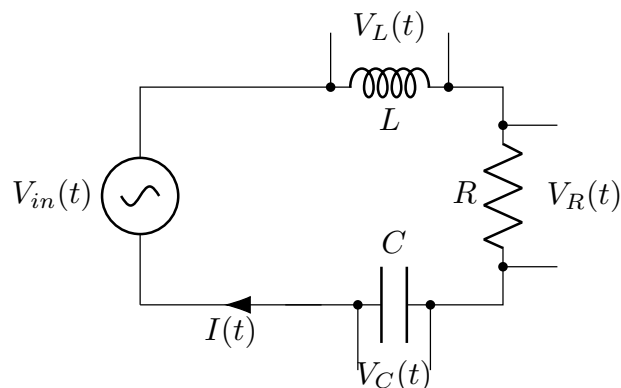
$I(t)$ is the current in the circuit in amps.
L is the inductance in henries.
R is the resistance in ohms.
C is the capacitance in farads.
$V_{in}(t)$ is the input voltage to the circuit.
$Q(t)$ is the charge on the capacitor, so $I(t) = Q'(t)$.

From physics we get that the voltage drops across each of the circuit elements are

$$V_L = LI' = LQ'', \quad V_R = RI = RQ', \quad V_C = \frac{Q}{C}. \tag{1}$$

The amazing thing is that this and Kirchhoff's voltage law (KVL) is all the physics we need to understand this circuit. The rest is linear constant coefficient DEs and complex arithmetic.

4.1 The basic differential equations

Kirchoff's voltage law (KVL) says the voltage drop around a closed loop is 0. The convention is that V_{in} gives the voltage gain in the direction of the current, while V_L, V_R, V_C all give voltage drops. So KVL implies

$$V_L + V_R + V_C = V_{in}.$$

Now using the given formulas for the voltage drops, we get the following differential equations.

$$LI' + RI + \frac{1}{C}Q = V_{in}(t) \tag{2}$$

$$LQ'' + RQ' + \frac{1}{C}Q = V_{in}(t) \quad \text{(Use } I = Q' \text{ in the first DE.)} \tag{3}$$

$$LI'' + RI' + \frac{1}{C}I = V_{in}(t)' \quad \text{(Differentiate the first DE and use } I = Q'.\text{)} \tag{4}$$

5 Complex impedance and Ohm's law for circuits

We make the crucial assumption of sinusoidal input (alternating current):

$$V_{in}(t) = V_0 \cos(\omega t).$$

In the following, we will add a tilde to indicate a complexified quantity, e.g., if $V_{in}(t) = V_0 \cos(\omega t)$, then $\tilde{V}_{in}(t) = V_0 e^{i\omega t}$.

First, we apply complex replacement to Differential Equation 4:

$$L\tilde{I}'' + R\tilde{I}' + \frac{1}{C}\tilde{I} = \tilde{V}'_{in}(t) = i\omega V_0 e^{i\omega t}. \tag{5}$$

This has characteristic polynomial $P(r) = Lr^2 + Rr + \frac{1}{C}$. So, the exponential response formula gives the solution

$$\tilde{I}(t) = \frac{i\omega V_0 e^{i\omega t}}{P(i\omega)} = \frac{i\omega V_0 e^{i\omega t}}{-L\omega^2 + Ri\omega + 1/C}.$$

Dividing both numerator and denonimator by $i\omega$ changes this formula to

$$\tilde{I}(t) = \frac{V_0 e^{i\omega t}}{iL\omega + R + 1/(iCw)} = G(i\omega)V_0 e^{i\omega t} = G(i\omega)\tilde{V}_{in}(t),$$

where $G(i\omega) = \dfrac{1}{iL\omega + R + 1/(iC\omega)}$ is the complex gain for the system.

Finally, the complex impedance of the system is

$$Z(\omega) = \frac{1}{G(i\omega)} = iL\omega + \frac{1}{iC\omega} + R. \tag{6}$$

Now, since $\tilde{I}(t) = G(i\omega)\tilde{V}_{in}(t)$, we have

$$\tilde{V}_{in}(t) = Z(\omega)\tilde{I}(t).$$

This is the complex version of Ohm's law: voltage = impedance $\times$ current. (Always assuming $\tilde{V}_{in} = V_0 e^{i\omega t}$.)

Looking at Equation 6 for impedance, we can associate a separate impedance to each element.

Impedance of inductor: $Z_L = iL\omega$ (depends on ω).

Impedance of resistor: $Z_R = R$ (independent of ω).

Impedance of capacitor: $Z_C = \dfrac{1}{iC\omega}$ (depends on ω).

Note:

1. The total complex impedance for a circuit wired in series is just the sum of the individual impedances. (Just like resistances in series.)

2. Complexifying the voltage drops in Equation 1, we see that the voltage drops across each element satisfy Ohm's law.

$\tilde{V}_L = L\tilde{I}' = Li\omega\tilde{I} = Z_L\tilde{I}$ (since $\tilde{I}(t) = G(i\omega)V_0 e^{i\omega t}$, so $\tilde{I}'(t) = i\omega\tilde{I}(t)$)

$\tilde{V}_R = R\tilde{I}$

$\tilde{V}_C = \dfrac{1}{C}\tilde{Q} = \dfrac{1}{C}\int \tilde{I}(t)\,dt = \dfrac{1}{iC\omega}\tilde{I} = Z_C\tilde{I}$ (again, using that $\tilde{I}(t)$ is a multiple of $e^{i\omega t}$).

6   Phasors

Phasor diagrams are a nice graphical way of presenting the voltages and currents associated with our LRC circuit. The word phasor essentially means $e^{i(\omega t - \phi)}$.

6.1   Simple complex arithmetic fact

You should be clear that in the complex plane multiplication by i is the same as rotation by $\pi/2$. Likewise, division by i is the same as rotation by $-\pi/2$.
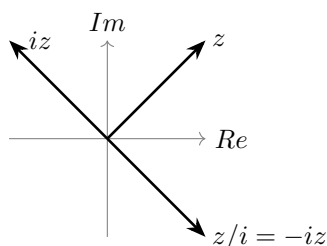
6.2   Amplitude-phase form

We start by writing the impedance for the system in Equation 4 in the form $R + iS$.

$$Z(w) = iL\omega + R + \frac{1}{iC\omega} = R + i(L\omega - \frac{1}{C\omega}) = R + iS, \quad \text{where } S = L\omega - \frac{1}{C\omega}.$$

In amplitude phase form $Z(\omega) = \sqrt{R^2 + S^2}e^{i\phi(\omega)}$, where $\phi(\omega) = \tan^{-1}(S/R)$.

Notice the sign of $\phi(\omega)$ is in the first or fourth quadrant depending on the sign of $S(\omega) = L\omega - 1/C\omega$. This implies that $\phi(\omega)$ is between $-\pi/2$ and $\pi/2$.

Complex Ohm's law says, $\tilde{V}_{in}(t) = Z(\omega)\tilde{I}(t) = |Z(\omega)|e^{i\phi(\omega)}\tilde{I}(t)$.

Since $\tilde{V}_{in}(t) = V_0 e^{i\omega t}$, we have

$$V_0 e^{i\omega t} = |Z(\omega)|e^{i\phi(\omega)}\tilde{I}(t) \quad \Rightarrow V_0 e^{i\omega - \phi(\omega)} = |Z(\omega)|\tilde{I}(t).$$

Taking real parts, we have $V_0 \cos(\omega t - \phi(\omega)) = |Z(\omega)|I(t)$, which is like Ohm's law, except with a phase shift.

The term $|Z(\omega)| = \sqrt{S^2 + R^2} = \sqrt{(L\omega - 1/C\omega)^2 + R^2}$ is called the real impedance of the system.

The term $S(\omega) = L\omega - 1/(C\omega)$ is called the reactance. As always, R is called the resistance.
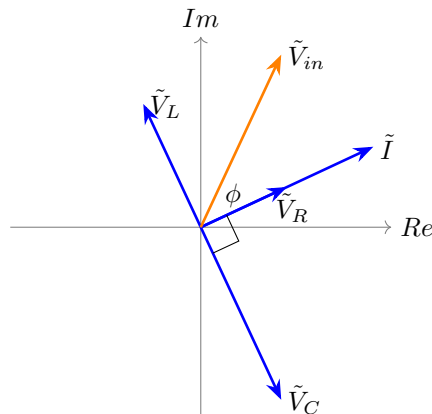
6.3   Phasor diagrams

To summarize the complex picture, we have input $\tilde{V}_{in}(t) = V_0 e^{i\omega t}$ and responses

$$\tilde{I}(t) = \frac{\tilde{V}_{in}(t)}{Z(\omega)} = \frac{V_0 e^{i\omega t - \phi(\omega)}}{|Z(\omega)|}, \quad \tilde{V}_L(t) = iL\omega\tilde{I}(t), \ \tilde{V}_R(t) = R\tilde{I}(t), \ \tilde{V}_C(t) = \frac{1}{iC\omega}\tilde{I}(t).$$

For any t, we can plot each of these as points in the complex plane. As t, changes, the entire picture rotates at frequency ω. The real values of the voltages are given by the x coordinates (the real parts) of their respective phasors.

Note: $\tilde{I}(t)$ is the phase angle $\phi(\omega)$ behind $\tilde{V}_{in}(t)$ (if ϕ is negative then $\tilde{I}$ is ahead of $\tilde{V}_{in}$).

Note: Because of the factor of i, $\tilde{V}_L$ is $\pi/2$ ahead of $\tilde{I}$. Likewise, $\tilde{V}_C$ is $\pi/2$ behind $\tilde{I}$ and $\tilde{V}_R$ is in phase with $\tilde{I}$.

Phasor diagram showing phase differences

There is a link below to the lovely Series LRC applet. It shows all of this and more!

6.4   Amplitude response and practical resonance

Natural frequency: If we set $R = 0$ and $V_{in} = 0$ in Equation 4, we see that the natural frequency of the system is $\omega_0 = 1/\sqrt{LC}$.

Practical Resonance: We know that the real gain is

$$\frac{1}{|Z(\omega)|} = \frac{1}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}.$$

Since the expression under the square root is a sum of squares, it is clearly minimized when the expression $L\omega - 1/(C\omega) = 0$. That is, when $\omega = 1/\sqrt{LC}$. This shows that ω_0 is the practical resonant frequency.

In terms of impedance, we get practical resonance when the reactance $S = Z_L + Z_C = L\omega - 1/(C\omega) = 0$.

In the phasor picture, at practical resonance, $\tilde{V}_{in}$, $\tilde{I}$ and $\tilde{V}_R$ all line up, i.e., the phase lag is 0 and $\tilde{V}_R = \tilde{V}_{in}$.

Practical resonance is a case where the corresponding sinusoidal graphs of the real voltages are neat enough to give a nice picture: the graph of V_R is exactly in phase with V_{in}; V_L and V_C have the same magnitude and are 180° out of phase; increasing R doesn't change V_R, but decreases the amplitude of I, V_L and V_C.

The applet 'Series LRC Circuit' (link is given below) shows all this beautifully.

7   Impedance in parallel

In this section we'll show that complex impedances in parallel combine like resistors in parallel. That is, if impedances Z_1 and Z_2 are in parallel then the total impedance of the pair, call it Z, satisfies $\dfrac{1}{Z} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$. We'll use this in the form

$$Z = \frac{1}{1/Z_1 + 1/Z_2}.$$
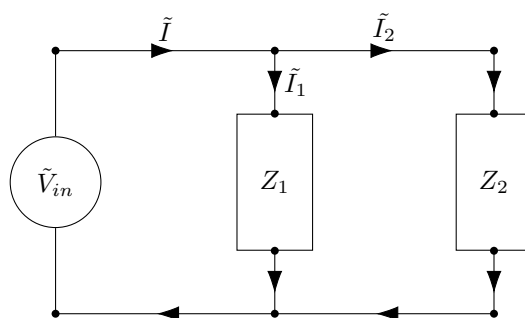
Circuit with impedances in parallel

To see this, we use the complex Ohm's law, KVL and Kirchhoff's current law (KCL). Referring to the figure above, KCL implies $\tilde{I} = \tilde{I}_1 + \tilde{I}_2$ and KVL implies $\tilde{V}_{in} = Z_1 \tilde{I}_1 = Z_2 \tilde{I}_2$. Thus, $\tilde{I}_1 = \dfrac{\tilde{V}_{in}}{Z_1}$ and $\tilde{I}_2 = \dfrac{\tilde{V}_{in}}{Z_2}$. So,

$$\tilde{I} = \tilde{I}_1 + \tilde{I}_2 = \frac{\tilde{V}_{in}}{Z_1} + \frac{\tilde{V}_{in}}{Z_2} = \tilde{V}_{in}\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) \quad \Rightarrow \quad \tilde{V}_{in} = \frac{1}{1/Z_1 + 1/Z_2}\tilde{I} = Z\tilde{I}. \quad \blacksquare$$

8   Series LRS applet

A nice applet showing all of this is at
https://mathlets.org/mathlets/series-rlc-circuit/.

Suggested applet exercise Set it to show you all four voltages and the current I. Set $L = 500$ mH, $C = 100\,\mu F$, $R = 250$ ohms.

Compute the resonant frequency of the system.

Move ω to the resonant frequency, watch the phasors and the sinusoidal plots as you do this.

With ω set at ω_0 watch the amplitudes of the 3 output voltages and the output current as R increases. Explain everything you see in terms of the complex Ohm's laws. (And the exponential response formula solution for $\tilde{I}$.)

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