Solutions Day 10, F 2/16/2024

Topic 5: Homogeneous, linear, constant coefficient DEs (day 2)

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Problem 1. *Consider the damped harmonic oscillator* $2x'' + 8x' + 6x = 0$.

(a) *What is the natural frequency of this oscillator?*

Solution: The natural frequency is the frequency of oscillation of the spring-mass system without damping. Without damping the equation is $2x'' + 6x = 0$. This has characteristic roots $\pm\sqrt{3}\,i$. That is, the solution is $x(t) = c_1 \cos(\sqrt{3}, t) + c_2 \sin(\sqrt{3}\,t)$. This oscillates at (angular) frequency $\boxed{\omega_0 = \sqrt{3}}$.

(b) *What type of damping (over, under or critical) does it have?*

Solution: Characteristic equation: $2r^2 + 8r + 6 = 0$.

Characteristic roots: $r = -1, -3$.

Negative real roots imply the system is overdamped.

(c) *How fast does it decay to 0?*

Solution: The general solution is $x(t) = c_1 e^{-t} + c_2 e^{-3t}$. This decays like the exponential with the largest exponent, i.e., like e^{-t}.

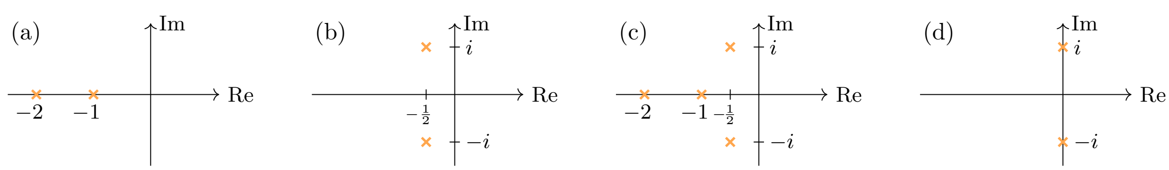
Problem 2. *Solve the DE* $2x'' + 8x' + 6x = 0$ *with initial conditions* $x(0) = 1$, $x'(0) = 0$. *(Hint: this is the same DE as in Problem 1.)*

Solution: From Problem 1, we know $x(t) = c_1 e^{-t} + c_2 e^{-3t}$. Using the initial conditions we get equations for c_1 and c_2:

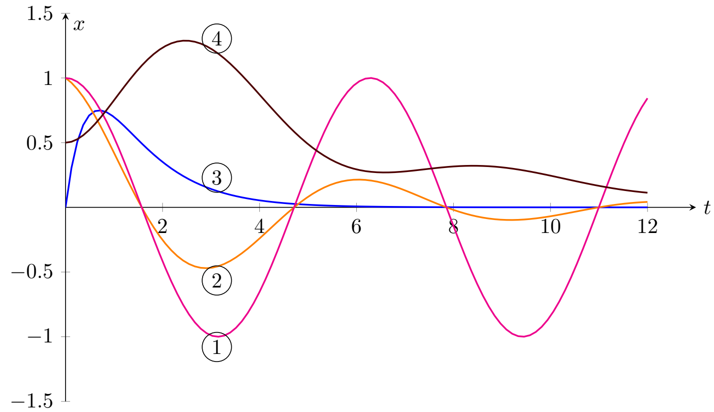
$$\begin{aligned} x(0) &= c_1 + c_1 = 1 \\ x'(0) &= -c_1 - 3c_2 = 0 \end{aligned}$$

Using any algebraic method you want: $c_1 = 3/2$, $c_2 = -1/2$, so $\boxed{x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}}$.

Problem 3. *Each of the following pole diagrams show the characteristic roots of a linear, constant coefficient, homogeneous DE.*



For each: Give the order of the system. Is it oscillatory? What is the exponential decay rate? Match the graphs below to the system.



Solution: System (a): This has two real negative roots.

Two roots implies the system is order 2.

Real roots implies the system is not oscillatory.

Roots are -1, -2. So the solution is $x = c_1 e^{-t} + c_2 e^{-2t}$. This decays like e^{-t}, i.e., like the slower decaying term, i.e., like the exponential associated with right-most root.

A second-order system with real, negative roots models an overdamped harmonic oscillator. From rest, i.e., where $x' = 0$ and the graph reaches a max or min, an overdamped oscillator must decay to 0 without oscillation and without crossing the t-axis. The only graph that fits this is Graph 3.

System (b): This has two complex roots with negative real part.

Two roots implies the system is order 2.

Complex roots implies the system is oscillatory.

Right most root has real part $-1/2$, so solutions decay like $e^{-t/2}$.

A second-order system with complex roots with negative real parts models an underdamped harmonic oscillator. This oscillates around $x = 0$ (equilibrium) with decaying amplitude. The only graph that fits this is Graph 2.

System (c): This has 2 negative real roots and 2 complex roots with negative real part.

Four roots implies the system is order 4.

Complex roots implies the system is oscillatory.

Right most root has real part $-1/2$, so solutions decay like $e^{-t/2}$.

Since solutions oscillate and decay and graph 2 is taken, this matches Graph 4.

System (d): This has 2 pure imaginary parts.

Two roots implies the system is order 2.

Imaginary roots implies the system is oscillatory.

All roots have real part 0 implies no decay, i.e., decays like e^{0t}.

This models an undamped (simple) harmonic oscillator. The graph of a solution is a sinusoid, i.e., $\boxed{\text{Graph 1}}$.

Problem 4. *Consider $z_1(t) = e^{(-2+3i)t}, \quad z_2(t) = e^{(-2-3i)t}$.*

(a) *Why are z_1, z_2 called complex-valued functions?*

Solution: Because $z_1(t)$ and $z_2(t)$ give complex numbers when evaluated at t.

(b) *Show that the (very special) linear combinations*

$$x_1 = \frac{1}{2} z_1 + \frac{1}{2} z_2, \quad x_2 = \frac{1}{2i} z_1 - \frac{1}{2i} z_2$$

are real valued.

Solution: By Euler's formula $\begin{cases} z_1 &= e^{-2t}(\cos(3t) + i\sin(3t)) \\ z_2 &= e^{-2t}(\cos(3t) - i\sin(3t)) \end{cases}$.

So, $x_1(t) = \frac{1}{2} z_1 + \frac{1}{2} z_2 = e^{-2t}\cos(3t)$. Likewise, $x_2(t) = \frac{1}{2i} z_1 - \frac{1}{2i} z_2 = e^{-2t}\sin(3t)$.

Both $x_1(t)$ and $x_2(t)$ produce real numbers for all values of t, hence they are real-valued.

Problem 5. *Suppose x_1, x_2 are solutions to $x'' + 8x' + 7x = 0$. Verify the superposition principle for linear, homogeneous DEs. That is, show that $x = c_1x_1 + c_2x_2$ is also a solution for all constants c_1, c_2.*

Solution: We plug x into the DE and chug through the algebra to show the equation holds.

$$\begin{aligned} x'' + 8x' + 7x &= (c_1x_1 + c_2x_2)'' + 8(c_1x_1 + c_2x_2)' + 7(c_1x_1 + c_2x_2) \\ &= c_1x_1'' + c_2x_2'' + c_18x_1' + c_28x_2' + c_17x_1 + c_27x_2 \\ &= c_1(\underbrace{x_1'' + 8x_1' + 7x_1}_{\substack{\text{0 by assumption} \\ \text{that } x_1 \text{ is a solution}}}) + c_2(\underbrace{x_2'' + 8x_2' + 7x_2}_{\substack{\text{0 by assumption} \\ \text{that } x_2 \text{ is a solution}}}) \\ &= 0 \end{aligned}$$

For $x = c_1x_1 + c_2x_2$, we have shown that $x'' + 8x' + 7x = 0$, i.e., that x is a solution to the DE. ■

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