Solutions Day 14, F 2/23/2024

Topic 6: Linearity of $P(D)$ and superposition (day 2)

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Problem 1. *Let $P(r) = r^2 + 4r + 3$*

(a) *Compute $P(D)e^{2t}$*

Solution: $P(D) = D^2 + 4D + 3I$, i.e., $P(D)x = x'' + 4x' + 3x$. So,

$$P(D)e^{2t} = 4e^{2t} + 8e^{2t} + 3e^{2t} = 15e^{2t}, \quad \text{i.e., } P(D)e^{2t} = P(2)e^{2t}.$$

(b) *Solve $P(D)x = e^{2t}$ by guessing a solution of the form $x = ce^{2t}$.*

Solution: Plug our guess into the DE $P(D)x = e^{2t}$. This gives

$$\begin{aligned} P(D)(ce^{2t}) &= c \cdot 15e^{2t} \quad \text{(from Part (a))} \\ &= e^{2t} \quad \text{(right side of the DE)} \end{aligned}$$

So, $c = 1/15$ and $x_p(t) = \dfrac{e^{2t}}{15}$.

(c) *Solve $P(D)x = e^{2t}$ by applying the ERF.*

Solution: ERF: $P(2) = 15$, so $x_p(t) = \dfrac{e^{2t}}{P(2)} = \dfrac{e^{2t}}{15}$. (Same as Part (b).)

(d) *Solve $P(D)x = e^{-3t}$ using the extended ERF.*

Solution: $P(-3) = 0$ –so we'll need the extended ERF.

$P'(r) = 2r + 4 \quad \Rightarrow P(-3) = -2$. Thus, $x_p(t) = \dfrac{te^{-3t}}{P'(-3)} = -\dfrac{te^{-3t}}{2}$.

Problem 2.

(a) *Solve $x'' + 2x' + 4x = \cos(3t)$ using the SRF.*

Solution: $P(r) = r^2 + 2r + 4$. So, $P(3i) = -9 + 6i + 4 = -5 + 6i$.

In polar form: $|P(3i)| = \sqrt{61}$, $\boxed{\phi = \text{Arg}(P(3i)) = \tan^{-1}(-6/5) \text{ in Q2}}$.

So, by the SRF, $\boxed{x_p(t) = \dfrac{\cos(3t - \phi)}{\sqrt{61}}}$.

(b) *Solve $x'' + 2x' + 4x = e^{-3t}\cos(2t)$ using complexification.*

Solution: Complex replacement:

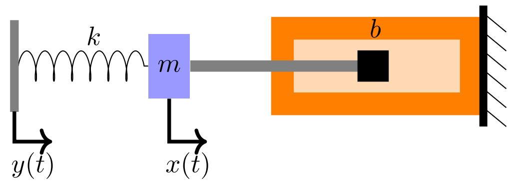
$$\begin{aligned} & x'' + 2x' + 4x = e^{-3t}\cos(2t) \\ \longrightarrow \quad & z'' + 2z' + 4z = e^{-3t}e^{2it} = e^{(-3+2i)t}, \quad x = \text{Re}(z). \end{aligned}$$

By the ERF, $z_p(t) = \dfrac{e^{(-3+2i)t}}{P(-3+2i)}$

$P(-3+2i) = ... = 3-8i$. So, $|P(-3+2i)| = \sqrt{73}$, $\boxed{\phi = \text{Arg}(P(-3+2i)) = \text{Arg}(3-8i) = \tan^{-1}(-8/3) \text{ in Q4}}$.

So, $z_p(t) = \dfrac{e^{(-3+2i)t}}{\sqrt{73}\, e^{i\phi}} = \dfrac{e^{-3t}}{\sqrt{73}} e^{i(2t-\phi)} \quad \Rightarrow \boxed{x_p(t) = \dfrac{e^{-3t}}{\sqrt{73}} \cos(2t - \phi)}$.

Problem 3. *A spring-mass-dashpot is driven by pushing on the spring. Suppose the input $y(t)$ gives the position of the end of the spring. Find a DE modeling the displacement $x(t)$ of the mass from equilibrium.*



Solution: The amount the spring is stretched is $x - y$. So the spring force on the mass is $-k(x - y)$.

The velocity of the damper is x'. So the damping force is $-bx'$.

Thus, by Newton's 2nd law, $mx'' = -k(x - y) - bx'$ or $\boxed{mx'' + bx' + kx = ky}$.