Day 15, M 2/26/2024

Topic 7: Polynomial input
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1 Agenda

- Finish SRF from yesterday
- General solutions to inhomogeneous DES
- Polynomial input: method of undetermined coefficients (optimism)
- Algebra with constant coefficient operators (FOIL works)
- Algebra with non-constant coefficient operators (FOIL fails)
- Existence and uniquenss theorem (only if time)

2 General solution to inhomogeneous linear DES

Example 1. Find the general solution to $x'' + 4x' + 3x = 5\cos(3t)$.

Solution: Step 1. Find a particular solution using the SRF.

$P(3i) = -6 + 2i$. So, $|P(3i)| = 6\sqrt{5}$, $\boxed{\phi = \text{Arg}(-6+2i) = \tan^{-1}(-1/3) \text{ in Q2}}$.

SRF: $\boxed{x_p(t) = \dfrac{5\cos(3t-\phi)}{6\sqrt{5}}}$.

Step 2: Find the general homogeneous solution, i.e., solve $x'' + 4x' + 3x = 0$.

Characteristic roots are $-1, -3$, so $\boxed{x_h(t) = c_1 e^{-t} + c_2 e^{-3t}}$.

Step 3: General solution by superposition: $\boxed{x(t) = x_p(t) + x_h(t)}$.

3 Polynomial input (method of undetermined coefficients)

Example 2. Solve $x'' + 4x' + 3x = 2t + 3$.

Solution: Input is a polynomial of degree 1.

So we try $x(t) = At + B$ (polynomial of same degree).

Plug our guess into the DE:

$$\begin{aligned} x &= At + B \\ x' &= A \\ x'' &= 0 \end{aligned}$$

So, $\underbrace{x_p'' + 4x_p' + 3x_p = 0 + 4(A) + 3(At+B)}_{\text{left side of DE}} = \underbrace{2t+3}_{\text{right side of DE}}$.

Clean this up: $3At + (4A + 3B) = 2t + 3$.

Equating coefficients we get:

$$\begin{array}{llll} \text{Coefficient of } t: & 3A & & = 2 \Rightarrow A = 2/3 \\ \text{Coefficient of } 1: & 4A & +3B & = 3 \Rightarrow B = 1/9 \end{array}$$

So a particular solution is $\boxed{x_p(t) = \frac{2}{3}t + \frac{1}{9}.}$

General homogeneous solution: $x_h(t) = c_1 e^{-t} + c_2 e^{-3t}$

General solution by superposition: $x(t) = x_p(t) + x_h(t) = \frac{2}{3}t + \frac{1}{9} + c_1 e^{-t} + c_2 e^{-3t}$.

4 Algebra of constant of coefficient operators

FOIL works: e.g., $(D - 3I)(D - 2I) = D^2 - 5D + 6I)$.

Example 3. Solve $(D - 3I)(D - 2I)x = 0$.

Solution: Characteristic equation: $P(r) = (r - 3)(r - 2) = 0$

Roots: $r = 3, 2$.

General solution: $x(t) = c_1 e^{3t} + c_2 e^{2t}$.

5 Algebra of non-constant coefficient operators

FOIL fails: Take care. Be systematic.

Example 4. Define T_1 and T_2 by $T_1 f = tf'$, $T_2 f = f'$.

What is T_1T_2? What is T_2T_1?

Solution: The question asks how the operators act on functions.

$$\left.\begin{array}{l} T_1T_2f = T_1(T_2f) = T_1(f') = tf'' \\ T_2T_1f = T_2(T_1f) = T_2(tf') = (tf')' = f' + tf'' \end{array}\right\} \text{Not the same. Order matters.}$$

6 Existence and uniqueness theorem

An initial value problem (IVP) is a DE with initial conditions.

Linear, second-order IVP in standard form:

$$\text{DE: } x'' + p(t)x' + q(t)x = f(t); \qquad \text{IC: } x(t_0) = b_0,\ x'(t_0) = b_1.$$

Existence and uniqueness theorem. If f, p, q are continuous, then the IVP has a unique solution.

Proof. Analysis – consider taking 18.100

Consequence. The general solution has two parameters.

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