Solutions Day 15, M 2/26/2024

Topic 7: Polynomial input

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Problem 1. *Find a particular solution to $2x'' + 3x' + 4x = 5t$.*

Solution: The input is a polynomial of degree 1, so we try a solution of the same degree: $x_p(t) = At + B$. Computing derivatives and substituting this into the DE we get:

$$\begin{array}{rcccc} x_p & = & At & + & B \\ x_p' & = & & & A \\ x_p'' & = & 0 & & \end{array}$$

So, $2x_p'' + 3x_p' + 4x_p = 3A + 4(At + B) = 4At + (3A + 4B) = 5t$.

Equating coefficients we get:

$$\begin{array}{lccl} \text{Coefficient of } t: & 4A & & = 5 \;\Rightarrow\; A = 5/4 \\ \text{Coefficient of } 1: & 3A & +4B & = 0 \;\Rightarrow\; B = -15/16 \end{array}$$

So, $\boxed{x_p(t) = \frac{5}{4}t - \frac{15}{16}.}$

Problem 2. *What solution would you guess for the following. (Do not solve.)*

(a) $2x''' + 3x'' + 4x' + 5x = 2t^2 + 3t + 4$

Solution: Try $x = At^2 + Bt + C$. (Same order as the input.)

(b) $x'' + t^2x' + 2x = e^{2t}$ *(Trick question!)*

Solution: This is not constant coefficient. We haven't learned how to make a good guess yet. (I might guess a power series.)

Problem 3. *Find the general solution to each of the following.*

(a) $(D - 3I)(D - 4I)(D - 5I)x = e^{2t}$

Solution: Characteristic polynomial: $P(r) = (r - 3)(r - 4)(r - 5)$.

Particular solution by ERF: $P(2) = (-1)(-2)(-3) = -6 \quad \Rightarrow \boxed{x_p(t) = \frac{e^{2t}}{P(2)} = -\frac{e^{2t}}{6}}.$

Homogeneous solution: Roots: 3, 4, 5 $\quad \Rightarrow \boxed{x_h(t) = c_1e^{3t} + c_2e^{4t} + c_3e^{5t}}.$

General solution: $\boxed{x(t) = x_p + x_h}.$

(b) $(D - 3I)(D - 4I)(D - 5I)x = \cos(\omega t)$.

Solution: Use the SRF: $x_p(t) = \dfrac{\cos(\omega t - \phi)}{|P(i\omega)|}$.

$P(i\omega) = (i\omega - 3)(i\omega - 4)(i\omega - 5) = -i\omega^3 + 12\omega^2 + 47i\omega - 60 = 12\omega^2 - 60 + i(-\omega^3 + 47\omega)$.

$$\boxed{|P(i\omega)| = \sqrt{(12\omega^2 - 60)^2 + (-\omega^3 + 47\omega)^2}, \quad \phi = \text{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{-w^3 + 47\omega}{12\omega^2 - 60}\right)}.$$

(The quadrant ϕ is in will vary as the two terms are positive or negative. A careful analysis would show that, ϕ is in Q2 for $0 < \omega < \sqrt{5}$; it is in Q1 for $\sqrt{5} < \omega < \sqrt{47}$ and in Q4 for $\omega > \sqrt{47}$.

As usual, $\boxed{x(t) = x_p + x_a}$, $x_h(t)$ as in Part (a).

Problem 4. *Let $T_1 f = f'$, $T_2 f = f^2$.*

(a) *Apply $T_1 T_2$ to a test function f.*

Solution: $T_1 T_2 f = T_1(f^2) = (f^2)' = 2ff'$.

(b) *Apply $T_2 T_1$ to a test function f.*

Solution: $T_2 T_1 f = T_2(f') = (f')^2$.

(c) *Do T_1 and T_2 commute.*

Solution: No. Clearly, $2ff' \neq (f')^2$ for most functions f.

Problem 5. *How many solutions does each IVP have?*

(a) $x'' + t^2 x' + 7x = e^{2t}$, $x(0) = 2$, $x'(0) = 3$.

Solution: Since t^2, 7, e^{2t} are all continuous, the existence and uniqueness theorem implies there is exactly one solution.

(b) $x'' + \frac{1}{t^2} x' + 7x = e^{3t}$, $x(0) = 2$, $x'(0) = 3$.

Solution: Since $1/t^2$ is not continuous at $t = 0$, the existence and uniqueness theorem does not apply. There might be 0, 1 or infinitely many solutions. We can't say without more work.

Problem 6. *Find a particular solution to $2x'' + 3x' = 5t$.*

Solution: Since the DE has an x' term, but no x term, we need to increase the powers of t in our usual guess by 1.

Try $x = At^2 + Bt$. So, $x' = 2At + B$, $x'' = 2A$.

This implies $2x'' + 3x' = 4A + 3(2At + B) = 5t$.

A bit of algebra: $6At + (4A + 3B) = 5t$.

Equate coefficients: $\left.\begin{array}{l} t: \ 6A = 5 \\ 1: \ 4A + 3B = 0 \end{array}\right\} A = 5/6, \ B = -10/9.$

So, $\boxed{x_p(t) = \frac{5}{6} t - \frac{10}{9}}$.

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