# Solutions Day 16, T 2/27/2024

Topic 8: Stability

Jeremy Orloff

**Problem 1.** *Say whether or not the systems $P(D)x = f$ with the given roots are stable.*

**(a)** *Roots:* $-1, -1, -2, -3$

**(b)** *Roots:* $-1, -1 \pm i, -2 \pm 3i$

**(c)** *Roots:* $0, -1, -2$

**(d)** *Roots:* $-1, -2, -3, 1 \pm i$

**Solution: (a)** All roots negative $\longrightarrow$ stable.

**(b)** All roots have negative real part $\longrightarrow$ stable.

**(c)** Roots $\leq 0$. This is an edge case. It is not asympotically stable. In some applications it my be stable "enough".

**(d)** Some roots have positive real part $\longrightarrow$ not stable.

**Problem 2.** *Are the following systems stable?*

**(a)** $3x'' + 4x' + 5x = f(t)$

**(b)** $x' + 2x = f(t)$

**(c)** $(D + 3I)(D - 3I)(D + 4I)x = f(t)$

**(d)** $x''' + 2x'' + 3x' + 7x = f(t)$

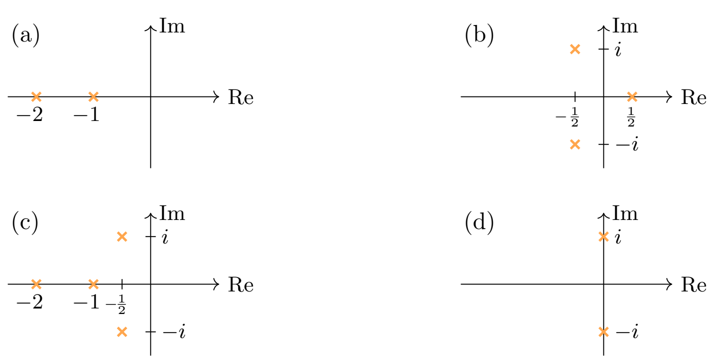
**Solution: (a)** Second-order, positive coefficients $\longrightarrow$ stable.

**(b)** First-order, $k > 0 \longrightarrow$ stable.

**(c)** Roots $-3$, $3$, $-4$. One positive root $\longrightarrow$ unstable.

**(d)** Third-order: $x'' + ax'' + bx' + cx = f(t)$, with $a = 2$, $b = 3$, $c = 7$. $a$, $b$, $c$ are positive, but $ab < c$. So, by the Routh-Hurwitz criteria, this is not stable.

**Problem 3.** *For a system $P(D)x = f$, the pole diagram shows the roots in the complex plane. Do the following pole diagrams come from stable systems?*




**Solution: (a)** All roots in the left half-plane (negative real part) $\longrightarrow$ stable,

**(b)** Some roots in the right half-plane $\longrightarrow$ unstable

**(c)** All roots in the left half-plane $\longrightarrow$ stable.

**(d)** Edge case. Not asymptotically stable. May be stable "enough" for some applications.

**Problem 4.** *Suppose $x = 8e^{2t}$ is a solution to $P(D)x = 5e^{2t}$, $x(0) = 8$.*

**(a)** *Give a solution to $P(D)x = 5e^{2(t-3)}$, $x(3) = 8$.*

**Solution:** Everything is shifted by 3. Since constant coefficient systems are time invariant, a solution is $x(t) = 8e^{2(t-3)}$.

**(b)** *Give a solution to $P(D)x = 10e^{2(t-3)}$, $x(3) = 16$.*

**Solution:** The input and initial condition are both twice that in Part (b). So a solution is $x(t) = 16e^{2(t-3)}$.

**(c)** *If $x_h(t) = c_1e^{-t} + c_2e^{-2t}$ solves $P(D)x_h = 0$, give a solution to $P(D)x = 0$, $x(3) = 1$, $x'(3) = -1$.*

**Solution:** The general solution to $P(D)x = e^{5t}$ is particular plus homogeneous:

$$x(t) = 8e^{2t} + c_1e^{-t} + c_2e^{-2t}$$

The computation will be easier if we change the meaning of $c_1$, $c_2$ and write this as

$$x(t) = 8e^{2t} + c_1e^{-(t-3)} + c_2e^{-2(t-3)}$$

Using the initial conditions, we get the equations

$$\begin{aligned} x(3) &= 8e^6 + c_1 + c_2 = 0 \\ x'(3) &= 16e^6 - c_1 - 2c_2 = 0 \end{aligned}$$

Solving, we find $c_1 = -32e^6$, $c_2 = 24e^6$. So,

$$x(t) = 8e^{2t} - 32e^6e^{-(t-3)} + 24e^6e^{-2(t-3)}.$$