Day 19, F 3/1/2024

Topic 9: Engineering language: input, gain, phase lag (day 2 of 3)

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1 Agenda

- Gain
- Resonance – pure and practical

2 Gain

Example 1. Consider $x' + kx = kf(t)$, where $k > 0$ and $f(t)$ is the input.

(a) Find the gain and phase lag.

(b) Show that this system is stable. Why does the stability of this system mean that, as t increases, we can ignore the homogeneous piece in the general solution?

(c) Why is this called a low pass filter?

Solution: (a) First, solve $x' + kx = B\cos(\omega t)$. $P(r) = r + k$. So,

$$P(i\omega) = i\omega + k \longrightarrow |P(i\omega)| = \sqrt{k^2+\omega^2}, \quad \boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^- 1(\omega/k) \text{ in Q1}}.$$

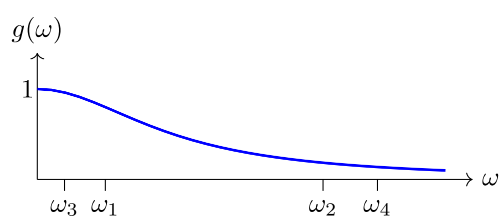
So, $x_p(t) = \dfrac{kB\cos(\omega t - \phi(\omega))}{|P(i\omega)|} = \dfrac{kB\cos(\omega t - \phi(\omega))}{\sqrt{k^2+\omega^2}}$.

Thus gain $= g(\omega) = \dfrac{k}{\sqrt{k^2+\omega^2}}$, phase lag $= \phi(\omega)$.

(b) Stability means the homogeneous solution decays to 0. Since the homogeneous solution is $x_h(t) = Ce^{-kt}$ (and $k > 0$), this is the case.

So every $x = x_p + x_h$ is asymptotically the same as x_p. That is, the $x_h(t)$ piece is transient and can be ignored as t increases.

(c) Below is the graph of $g(\omega)$. We see that $g(0) = 1$ and $g(\omega)$ is decreasing and goes to 0.



For low frequencies, say $\omega < \omega_1$, we see $g(\omega) \approx 1$.

For high frequencies, say $\omega > \omega_2$, we see $g(\omega) \approx 0$.

So, for example, input $f(t) = e\cos(\omega_3 t) + \cos(\omega_4 t)$ has output

$$x(t) = g(\omega_3)\cos(\omega_3 t - \phi(\omega_3)) + g(\omega_4)\cos(\omega_4 t - \phi(\omega_4)).$$

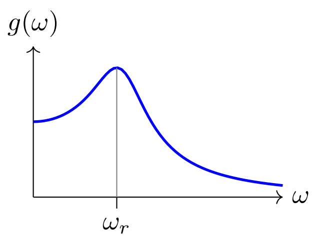
Since $g(\omega_4) \approx 0, \quad x(t) \approx g(\omega_3)\cos(\omega_3 t - \phi(\omega_3))$.

The system has "filtered out" the high frequency term $\cos(\omega_4 t)$ from the input and "passed" the low frequency term $\cos(\omega_3 t)$. So it is a "low pass filter".

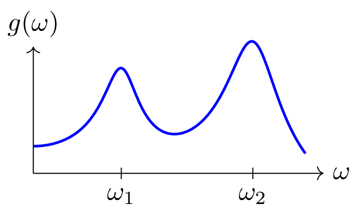
3 Practical resonance

A practical resonant frequency is a relative maximum of $g(\omega)$ for $\omega > 0$. ($\omega = 0$ doesn't count.) That is, it is a frequency that has a relatively large response.

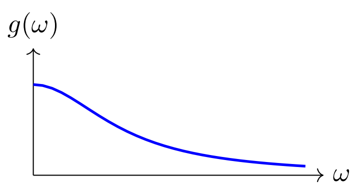
Graphically:



w_r is a practical resonant frequency.



$\omega_1, \quad \omega_2$ are practical resonant frequencies.



A system with no practical resonant frequencies.

Calculus: Practical resonance occurs at critical points of $g(\omega)$, i.e., where $g'(\omega) = 0$. (Also need to check the critical point is a relative maximum.)

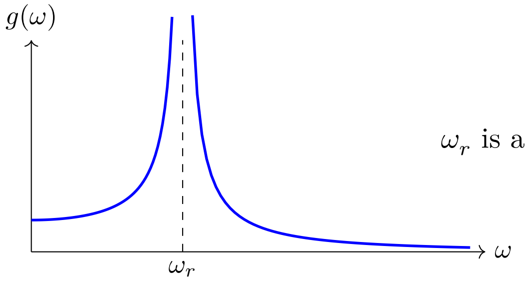
– Examples in today's problems.

4 Pure resonance

A pure resonant frequency is at a vertical asymptote in the gain graph.

– We say, it is where $g(\omega) = \infty$.

Graphically:



ω_r is a pure resonant frequency.

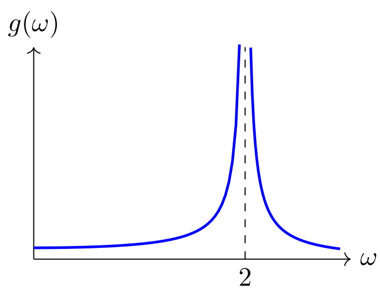
Example 2. Consider $x'' + 4x = B\cos(\omega t)$, where $B\cos(\omega t)$ is the input. Find the gain and any pure resonant frequencies.

Solution: $P(r) = r^2 + 4$. So, $P(i\omega) = 4 - \omega^2$.

For $\omega \neq 2$, , $|P(i\omega)| = |4 - \omega^2| \neq 0$, $\phi(\omega) = \text{Arg}(P(i\omega))$.

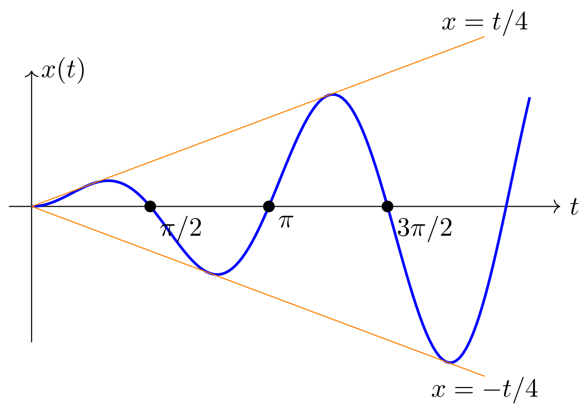
So, $x_p(t) = \frac{B\cos(\omega t - \phi(\omega))}{|4-\omega^2|}$.

The gain is $g(\omega) = \dfrac{1}{|4 - \omega^2|}$. This has a vertical asymptote at $\omega = 2$. That is "$g(2) = \infty$"



For $\omega = 2$, the extended SRF gives $x_p(t) = \dfrac{t}{4}\cos(2t - \pi/2)$.

The graph shows an oscillation with growing amplitude. This is the meaning of "gain $g(2) = \infty$".



Note: $\omega = 2$ is the natural frequency of the oscillator.