Solutions Day 19, F 3/1/2024

Topic 9: Engineering language: input, gain, phase lag, frequency response (day 2 of 3)

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Problem 1. *Consider the system* $x'' + x' + 4x = 4\cos(5t)$.

(a) *Find the sinusoidal response.*

Solution: Use the SRF: $P(5i) = -21+5i$. So, $|P(5i)| = \sqrt{466}$, $\boxed{\phi = \text{Arg}(P(5i)) = \tan^{-1}(-5/21) \text{ in Q2}}$.

So the sinusoidal response is $\boxed{x_p(t) = \dfrac{4\cos(5t-\phi)}{\sqrt{466}}}$.

(b) *Assume* $\cos(5t)$ *is the input. What are the gain and phase lag?*

Solution: Input amplitude $= 1$; output amplitude $= \dfrac{4}{\sqrt{466}}$. So, gain $= \boxed{\dfrac{4}{\sqrt{466}}}$. $\boxed{\text{Phase lag} = \phi}$.

(c) *Assume* $4\cos(5t)$ *is the input. What are the gain and phase lag?*

Solution: Input amplitude $= 4$. Output amplitude $= \dfrac{4}{\sqrt{466}}$.

So, gain $= \boxed{\dfrac{1}{\sqrt{466}}}$. (Always: output amplitude = gain × input amplitude.)

Phase lag $= \boxed{\phi}$

Problem 2. *A system is modeled by* $x' + kx = kf(t)$, $k > 0$. *We consider* $f(t)$ *to be the input.*

(a) *Solve the DE with input* $B\cos(\omega t)$.

Solution: We use the SRF. $P(r) = r + k$, so $P(i\omega) = i\omega + k$. Thus,

$|P(i\omega)| = \sqrt{k^2+\omega^2}$, $\boxed{\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}(\omega/k) \text{ in Q1}}$. So, $\boxed{x_p(t) = \dfrac{Bk\cos(\omega t - \phi(\omega))}{\sqrt{k^2+w^2}}}$.

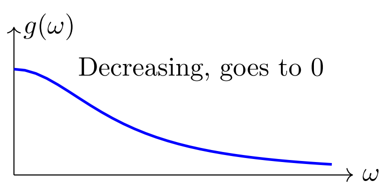
(b) *What are the gain and phase lag for the system?*

Solution: Input amplitude $= B$. Output amplitude $= \dfrac{Bk}{\sqrt{k^2+\omega^2}}$.

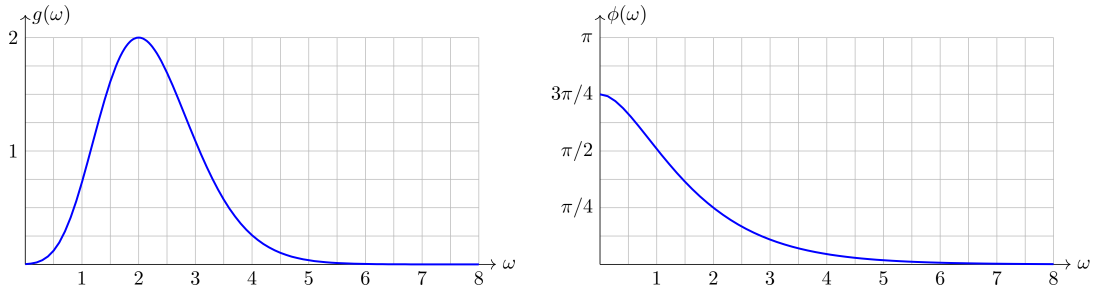
So, gain $= \dfrac{\text{output amp.}}{\text{input amp.}} = \dfrac{k}{\sqrt{k^2+\omega^2}}$ $\boxed{\text{Phase lag} = \phi(\omega)}$.

(c) *Graph the gain. (Be sure to label your axes.)*

Solution: Here is the gain plot. It is decreasing and goes asymptotically to 0.



Problem 3. *A system modeled by a constant coefficient, linear DE has gain and phase lag as shown.*



(a) *If the input is $B\cos(2t)$, what is the periodic repsonse?*

Solution: Since the input frequency is $\omega = 2$, the output is

$$x_p(t) = g(2)B\cos(2t - \phi(2)).$$

From the graphs: $g(2) = 2$, $\phi(2) = \pi/4$. So, $\boxed{x_p(t) = 2B\cos(2t - \pi/4)}$.

(b) *If the input is $3\cos(2t) + 3\cos(6t) + 3\cos(8t)$, give a good approximation to the response.*

Solution: Using superposition, the solution is

$$x_p(t) = g(2) \cdot 3\cos(2t - \phi(2)) + g(6) \cdot 3\cos(6t - \phi(6)) + g(8) \cdot 3\cos(8t - \phi(8)).$$

Since $g(6)$ and $g(8)$ are very close to 0, we have

$$x_p(t) \approx g(2) \cdot 3\cos(2t - \phi(2) = 6\cos(2t - \pi/4).$$

(c) *What input frequency has the biggest response?*

Solution: The graph shows the maximum gain is at $\omega = 2$. (This is called a practical resonant frequency.)

Problem 4. *Find all the resonant frequencies of the following systems.*

(a) $x'' + x' + 9x = f(t)$, $f(t) =$ *input.*

Solution: First, we need the gain for $x'' + x' + 9x = \cos(\omega t)$.

Since $x_p(t) = \dfrac{\cos(\omega t - \phi(\omega))}{|P(i\omega)|}$ and input amplitude is 1, gain $= g(\omega) = \dfrac{1}{|P(i\omega)|}$.

Now, $P(i\omega) = 9 - \omega^2 + i\omega \quad \Rightarrow |P(i\omega)| = \sqrt{(9-\omega^2)^2 + \omega^2}$.

Resonances are at relative maxima of $g(\omega)$. These are the same as the relative minima of $h(\omega) = \dfrac{1}{g(\omega)^2} = (9-\omega^2)^2 + \omega^2$, i.e., we need to find where $h'(\omega) = 0$. That is,

$$h'(\omega) = -4\omega(9 - \omega^2) + 2\omega = 0 \quad \Rightarrow \omega = 0 \text{ or } 2\omega^2 = 17 \quad \Rightarrow \omega = 0 \text{ or } \omega = \sqrt{17/2}.$$

We require $\omega > 0$, so the only practical resonant frequency is $\omega = \sqrt{17/2}$.

Technically, we should show this is a relative maximum. This is easy using the second derivative, or geometrically since $g(\sqrt{17/2}) > g(0)$ and $g(\omega)$ goes to 0 as ω gets large.

(b) *$x'' + 8x' + 7x = f(t)$, $f(t)$ = input.*

Solution: Again, gain is $g(\omega) = \dfrac{1}{|P(i\omega)|} = \dfrac{1}{\sqrt{(7-\omega^2)^2 + 64\omega^2}}$.

We can be a bit clever by noticing that relative maxima for $g(\omega)$ are at the same values of ω as minima of $\dfrac{1}{g(\omega)^2} = (7-\omega^2)^2 + 64\omega^2$.

Call this $h(\omega)$ and compute $h'(\omega) = -4\omega(7-\omega^2) + 128\omega = 0$.

Factoring we get $4\omega\left[-(7-\omega^2) + 32\right] = 0$. With a little algebra, we get $\omega = 0$ or $\omega = \pm\sqrt{15}\, i$.

Since we demand real, positive resonantfrequences, this system has no practical resonant frequency.

(c) *$x'' + 8x' + 7x = f'(t)$, $f(t)$ = input.*

Solution: Here is another useful trick for doing these computations: We complexify before taking the derivative. The complexified system is

$$z'' + 8z' + 7z = (e^{i\omega t})' = i\omega e^{i\omega t}, \quad x = \mathrm{Re}(z).$$

The ERF gives $z_p(t) = \dfrac{i\omega e^{i\omega t}}{P(i\omega)} = \dfrac{\omega e^{i\pi/2} e^{i\omega t}}{|P(i\omega)| e^{i\phi(\omega)}} = \dfrac{\omega e^{i(\omega t + \pi/2 - \phi(\omega))}}{|P(i\omega)|}$.

(As usual, $\phi(\omega) = \mathrm{Arg}(P(i\omega))$ –which doesn't play a role in the gain.)

We have, $x_p(t) = \mathrm{Re}(z_p) = \dfrac{\omega\cos(\omega t - (\phi(\omega) - \pi/2))}{\sqrt{(7-\omega^2)^2 + 64\omega^2}}$. So the gain is $g(\omega) = \dfrac{\omega}{|P(i\omega)|} = \dfrac{\omega}{\sqrt{(7-\omega^2)^2 + 64\omega^2}}$.

We could find relative maxima by solving $g'(\omega) = 0$, but in this case there is a nice shortcut.

$$g(\omega) = \frac{\omega}{\sqrt{(7-\omega^2)^2 + 64\omega^2}} = \frac{1}{\sqrt{\frac{(7-\omega^2)^2 + 64\omega^2}{\omega^2}}} = \frac{1}{\sqrt{\left(\frac{7-\omega^2}{\omega}\right)^2 + 64}}.$$

This has a maximum when the denominator is as small as possible. Because the square term is positive, the denominator is smallest when

$$\frac{(7-\omega^2)}{\omega} = 0, \quad \text{i.e., when } \boxed{\omega = \sqrt{7}}.$$

Problem 5. *Consider the system $2x'' + 8x = f'(t)$, where $f(t)$ is considered the input.*

(a) *Find the periodic response to $f(t) = B\cos(\omega t)$, for all ω.*

Solution: DE: $2x'' + 8x = (B\cos(\omega t))'$.

Complex replacement: $\quad 2z' + 8z = \left(Be^{i\omega t}\right)' = Bi\omega e^{i\omega t}, \quad x = \mathrm{Re}(z)$.

$P(i\omega) = 8 - 2\omega^2$. We see that $P(i\omega) \neq 0$, for $\omega \neq 2$.

Thus, for $\omega \neq 2$, the regular ERF gives us

$$z_p(t) = \frac{Bi\omega e^{i\omega t}}{P(i\omega)} = \frac{Be^{i\pi/2}\omega e^{i\omega t}}{|P(i\omega)|e^{i\phi(\omega)}}, \text{ where } \phi(\omega) = \text{Arg}(P(i\omega)).$$

So, $z_p(t) = \dfrac{B\omega}{|P(i\omega)|}\, e^{i(\omega t - \phi + \pi/2)} \quad \Rightarrow x_p(t) = \text{Re}(z_p) = \dfrac{B\omega}{|P(i\omega)|}\cos(\omega t - (\phi - \pi/2))$.

Since $P(i\omega) = 8 - 2\omega^2$ is real, its argument is either 0 or π. So, for $\omega \neq 2$, we have

$$|P(i\omega)| = |8 - 2\omega^2| \quad \text{and} \quad \phi(\omega) = \begin{cases} 0 & \text{if } 0 < \omega < 2 \\ \pi & \text{if } \omega > 2 \end{cases}.$$

For $\omega \neq 2$ we have

$$\boxed{x_p(t) = \frac{B\omega\cos(\omega t - (\phi(\omega) - \pi/2))}{|P(i\omega)|} = \begin{cases} \frac{B\omega\cos(\omega t + \pi/2)}{|8-2\omega^2|} & \text{if } 0 < \omega < 2 \\ \frac{B\omega\cos(\omega t - \pi/2)}{|8-2\omega^2|} & \text{if } \omega > 2 \end{cases}}$$

$\omega = 2$ is a special case, requiring the extended ERF.

The complex equation is $2z'' + 8z = Bi2e^{i2t}, \quad x = \text{Re}(z)$.

Since $P(2i) = 0$, the extended ERF gives $z_p(t) = \dfrac{tBi2e^{2it}}{P'(2i)}$.

$P'(r) = 4r$, so $P'(2i) = 8i = 8e^{i\pi/2}$. Thus,

$$z_p(t) = \frac{Bi2te^{i2t}}{P'(2i)} = \frac{B2te^{i\pi/2}}{8e^{i\pi/2}} = \frac{Be^{i2t}}{4}.$$

Finally, $\boxed{x_p(t) = \text{Re}(z_p) = \dfrac{Bt\cos(2t)}{4}, \quad \text{for } \omega = 2}$.

(b) *Give formulas for the gain and phase lag.*

Solution: From Part (a), we have for $\omega \neq 2$,

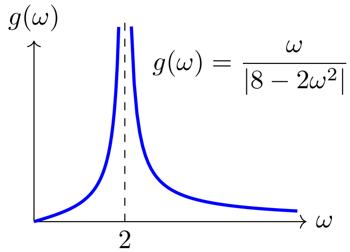
$$\text{gain } \; g(\omega) = \frac{\omega}{|8 - 2\omega^2|}$$

$$\text{phase lag } \; \phi(\omega) = \begin{cases} -\pi/2 & \text{for } 0 < \omega < 2 \\ \pi/2 & \text{for } \omega > 2 \end{cases}$$

There is no periodic solution when $\omega = 2$, so, officially, the gain is not defined. Since $g(\omega)$ goes to infinity at $\omega = 2$, we will say that $g(2) = \infty$.

(c) *Plot the gain.*

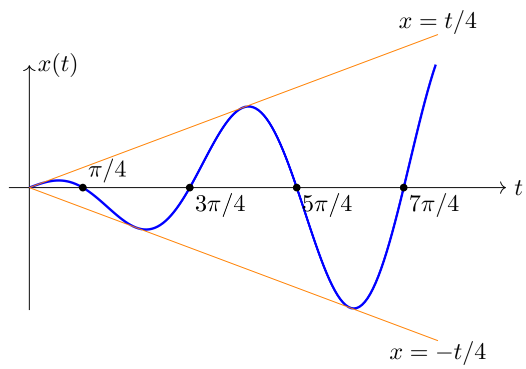
Solution: Here is the plot of $g(\omega) = \dfrac{\omega}{|8 - 2\omega^2|}$.



Note: $g(0) = 0$, $g(\omega)$ decays to 0 as $\omega \to \infty$, vertical assymptote at $\omega = 2$.

(d) *Plot the response when $\omega = 2$. Why do we say $g(2) = \infty$.*

Solution: For this graph, we'll set $B = 1$. The response is $x_p(t) = \dfrac{t\cos(2t)}{4}$.



There are two good reasons to say $g(2) = \infty$:

First, $g(\omega) = \dfrac{\omega}{|8 - 2\omega^2|}$. As $\omega \to 2$, we have $g(\omega) \to \infty$. That is, graphically, $g(\omega)$ has a vertical asymptote at $\omega = 2$.

Second, the graph of the solution shows an oscillation with amplitude increasing to ∞.