Day 2, T 2/6/2024

Finish Topic 1
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1 Agenda

- For tomorrow: Read Topic 2 notes
- Recap physical examples
- Our most important DE
- 2^{nd} most important DE
- Checking solutions
- Constructing models using Δx, Δt
- Separable DEs (in problems)
- Geometric problems (in problems)

2 Physical examples

Newton's law of cooling:	$x' = -k(x - E)$	$\Leftrightarrow$	$x' + kx = kE.$
Hooke's law:	$mx'' = -kx$	$\Leftrightarrow$	$mx'' + kx = 0.$
Exponential growth and decay:	$y' = ay$	$\Leftrightarrow$	$y' - ay = 0.$

3 The most important DE

$$\frac{dy}{dt} = ay, \qquad a \text{ a constant (parameter)} \qquad (****)$$

– Says the rate y changes is proportional to y.

– Solution: $y(t) = Ce^{at}$, C and arbitrary constant. (Will derive this in the problems.)

Checking that $y(t) = Ce^{at}$ is a solution to (****):

$$\left.\begin{array}{rl} \text{Left side of (****):} & y' = aCe^{at} \\ \text{Right side of (****):} & ay = aCe^{at} \end{array}\right\} \text{Same } \checkmark$$

(****) models:

- unconstrained population growth
- radioactive decay
- money in an interest bearing account
- initial spead of a disease

4 Second most important DE

$$\frac{dm''}{dx} + kx = 0, \qquad (\Leftrightarrow \; mx'' = -kx, \quad \text{Hooke's law}) \qquad (**)$$

Solution: $x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right).$

This is a two parameter family of solutions.

We will see how to derive this many times in the next few weeks.

5 Order of a DE = highest derivative in the DE

Example 1. $y' = ay$: 1^{st} order $\longrightarrow$ one parameter in the solution.

$mx'' + kx = 0$: 2^{nd} order $\longrightarrow$ two parameters in the solution.

6 Modeling a DE using small changes

(See Section 1.7 in Topic 1 notes.)

Example 2. $P(t)$ = a population of bacteria

$$\beta = 2\%/\text{hour} = \text{birth rate} \qquad (\beta = \text{Greek beta})$$
$$\delta = 1\%/\text{hour} = \text{death rate} \qquad (\delta = \text{Greek delta})$$

Model $P(t)$ with a differential equation.

Solution: Pick a small interval of time $[t,\, t + \Delta t]$.

Let ΔP = change in P over this interval = births - deaths.

Over a small time interval, P is roughly constant, so

$$\begin{array}{lcl} \text{births} & \approx & \beta\, P(t)\, \Delta t \qquad (\text{units: } \tfrac{1}{\text{hours}} \cdot \text{bacteria} \cdot \text{hours} = \text{bacteria}) \\ \text{deaths} & \approx & \delta\, P(t)\, \Delta t. \end{array}$$

So, $\Delta P \approx (\beta - \delta) P\, \Delta t \quad \longrightarrow \quad \dfrac{\Delta P}{\Delta t} \approx (\beta - \delta) P.$

In the limit, as $\Delta t \to 0$, this becomes exact.

$$\frac{dP}{dt} = (\beta - \delta) P.$$

Amazingly, the same logic holds if β, δ, depend on t or P. (As long as they are continuous.)

7 Reminders

- In separable equations, don't forget lost solutions.
- Since slope $= \dfrac{dy}{dx}$, problems involving graphs can lead to DEs.