Day 23, R 3/7/2024

Topic 10: Graphical methods: Direction fields, integral curves

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1 Agenda

- Integral curves
- Direction fields (slope fields)
- Method of isoclines
- Existence and uniqueness theorem
- Fences, funnels, squeezing

2 Introduction

Topics 10-12 are a unit on nonlinear first-order DEs: $\dfrac{dy}{dx} = f(x, y)$ or $\dfrac{dy}{dt} = f(t, y)$.

We'll do graphical, qualitative and numerical analysis.

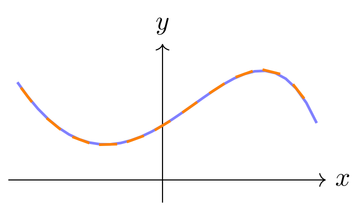
3 Running example

We'll keep a running example using the first-order DE:

$$y' = y - x^2 = f(x, y). \tag{1}$$

Definition: An integral curve is the graph of a solution to the DE.

At points on the integral curve we can draw little slope elements whose slope is that of the curve.



Integral curve for $y' = y - x^2$ with slope elements

Key point: we don't need the curve to draw the slope elements: For example on the curve through $(1, 2)$, we know the slope is $y' = f(1, 2) = 1$, so we could draw a slope line of slope 1 through $(1, 2)$.

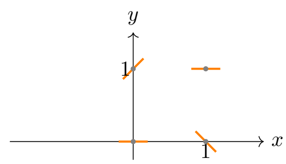
4 Direction (slope) field

At each point (x, y) put a little line segment of slope $f(x, y)$.

Example 1. Draw some slope elements for $y' = y - x^2$.

Solution: At $(0,0)$: $y' = 0$, at $(1,0)$: $y' = -1$, at $(0,1)$: $y' = 1$, at $(1,1)$: $y' = 0$.

Here are these four slope elements drawn in the plane.



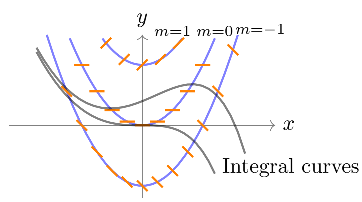
Exhausting if you aren't a computer!

4.1 Isoclines (method for drawing direction fields by hand)

Example 2. Draw some isoclines for $y' = y - x^2$.

Solution: Pick a slope m. Draw the curve where $y' = m$. Along the curve draw a number of slope elements of slope m.

So, $y' = y - x^2 = m$ or $y = x^2 + m$ = isocline with slope elements of slope m.



A few isoclines for $y' = y - x^2$.

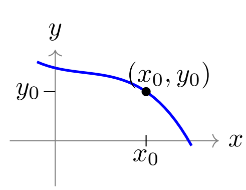
Do Problem 1.

5 Existence and Uniqueness Theorem

If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous, then the DE with initial conditions:

$$y' = f(x, y), \qquad y(x_0) = y_0$$

has exactly one solution. That is, the solution exists and is unique.



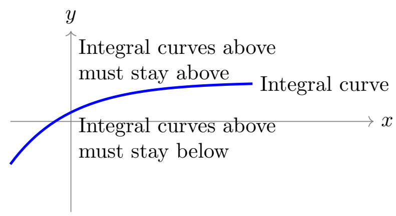
Exactly one integral curve through the point (x_0, y_0).

Takeaway: Integral curves don't cross (under mild assumptions).

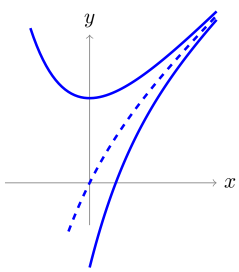
Do Problem 2

6 Fences and funnels

Integral curves are "fences": Because integral curves don't cross, any integral curve acts as a fence which other integral curves can't get past.

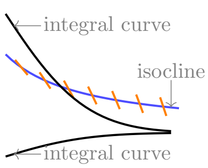


Two integral curves can form a "funnel".



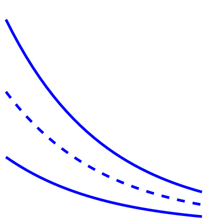
The outside (solid blue) integral curves form a funnel. Any integral curve that starts between them must stay there. So they are all funneled asymptotically to the same place.

Isoclines can act as one sided (upper or lower) fences.

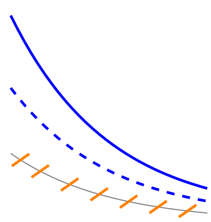


The isocline shown is an upper fence: An integral curve can cross from above to below. But one can't cross from below to above. That is, to the integral curve below it looks like a fence.

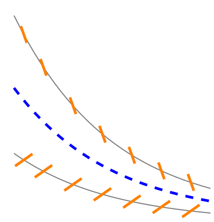
Funnels can have fences made from isoclines or integral curves.



Funnel from 2 integral curves



Funnel from integral curve and isocline as lower fence



Funnel from 2 isoclines: one a lower and one upper fence