Day 24, F 3/8/2024

Topic 11: Numerical methods for $y' = f(x, y)$

Jeremy Orloff

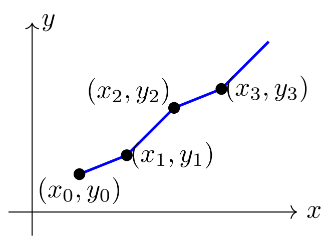
1 Agenda

- Euler's method
- 2nd derivative and concavity
- Numerical issues
- Error size

2 General approach

Problem: Given $y' = f(x, y)$, $y(x_0) = y_0$, estimate $y(x)$.

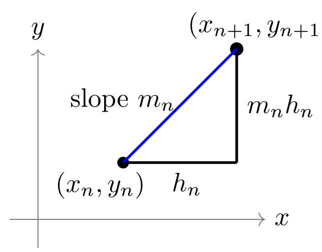
Main idea: We take steps: Start at (x_0, y_0), step to (x_1, y_1), step to $(x_2, y_2), \ldots$. We hope the resulting polygon approximates the integral curve which starts at (x_0, y_0).



All of our algorithms have a common structure.

To step from (x_n, y_n) to (x_{n+1}, y_{n+1}), we use

$x_{n+1} = x_n + h_n$
$y_{n+1} = y_n + m_n h_n$
The algorithm chooses h_n, m_n at each step.
h_n is called the stepsize at step n.
m_n is called the slope at step n.



3 Euler's method

Fixed stepsize: Choose one stepsize h for all steps.

$m_n = f(x_n, y_n)$ = slope of the direction field

Example 1. Consider $y' = x^2 y$, $y(-2) = 1$. Use Euler's method to estimate $y(-1)$ in 2 steps.

Solution: 2 steps to go from $x = -2$ to $x = -1$ implies stepsize $\boxed{h = 0.5}$

Make a table:

n	x_n	y_n	m_n	$m_n h$
0	-2	1	4	2
1	-1.5	3	6.75	3.375
2	-1	6.375		

Our estimate is $y(-1) \approx 6.375$.

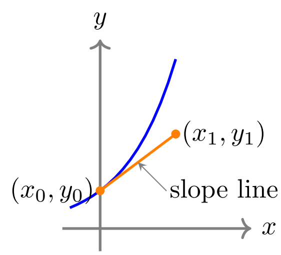
Smaller h would mean more steps (more computation), but would be more accurate.

Euler's method has linear error: cut h in half $\longrightarrow$ roughly cut the error in half.

4 Second derivative

Is the Euler estimate in the previous example and over or underestimate?

Idea: if the integral is concave up, then Euler underestimates.



Solution: We have $y' = x^2 y$. So, $y'' = 2xy + x^2 y'$ $\longleftarrow$ (Don't forget y is a function of x. Also, don't substitute for y', it gets too messy.)

We have $y(-2) = 1 \longrightarrow y'(-2) = (-2)^2 \cdot 1 = 4 \longrightarrow y''(-2) = 2(-2)(1) + (-2)^2(4) = 12$.

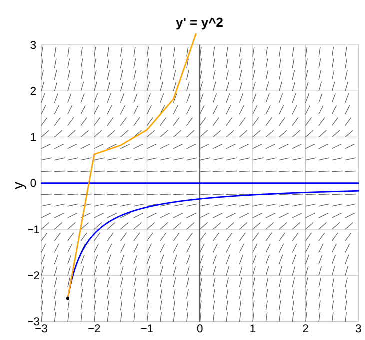
Since $y''(-2) > 0$, the integral curve is concave up at $x = -2$.

Conclusion: The first step is probably an underestimate. Most likely, so is the second step.

5 What can go wrong

5.1 Stepping across a boundary

We hope the Euler polygon will follow the integral curve starting at (x_0, y_0). But too big a step can cross a separatrix and go very wrong. Here, the blue integral curve starting at $(-2.5, -2.5)$ goes asymptotically to 0. The integral curve $y = 0$ is a separatrix. The curves below it go to 0 and those above go to infinity. With a big enough step, the orange Euler polygon steps across the separatrix and then goes to infinity instead of 0.



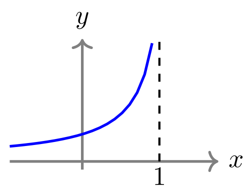
Rule of thumb: Try more than one stepsize h.

- Pick h and compute the estimate.
- Cut h in have and compute the estimate.
- Repeat this till the answer settles down.

5.2 Looking for something that's not there

Example 2. Consider $y' = y^2$, $y(0) = 1$.

We can solve this exactly: $y(x) = \dfrac{1}{1-x}$, valid on $(-\infty, 1)$. So, $y(1) =$" ∞".



Now, use Euler to estimate $y(1)$.

$$\begin{array}{lcl} n = 5 \text{ steps} & \longrightarrow & y(1) \approx y_5 = 4.11 \\ n = 10 \text{ steps} & \longrightarrow & y(1) \approx y_{10} = 6.13 \\ n = 50 \text{ steps} & \longrightarrow & y(1) \approx y_{50} = 18.13 \\ n = 100 \text{ steps} & \longrightarrow & y(1) \approx y_{100} = 30.39 \\ n = 500 \text{ steps} & \longrightarrow & y(1) \approx y_{500} = 108.82 \end{array}$$

This doesn't settle down. Following the rule of thumb will reveal this.

Lesson: Algorithms will compute. We need to think!

6 Other methods

- Improved Euler (RK2), Runga-Kutta (RK4, RK5, ...)
- Variable step size methods: choose h at each step.

These methods are more accurate than Euler's, but require more computation.