Solutions Day 24, F 3/8/2024

Topic 11: Numerical methods for $y' = f(x,y)$, $y(x_0) = y_0$

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Problem 1. *Let $y' = x - y, \quad y(1) = 3$.*

(a) *Use Euler's method to estimate $y(2)$ using 3 steps.*

Solution: Make a table using $h = \dfrac{2-1}{3} = \dfrac{1}{3}$.

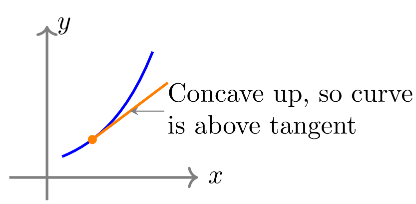
j	x_j	y_j	m_j	$m_j \cdot h$
0	1	3	-2	$-2/3$
1	$4/3$	$7/3$	-1	$-1/3$
2	$5/3$	2	$-1/3$	$-1/9$
3	2	$\boxed{17/9}$		

Estimate: $y(3) \approx 17/9$.

(b) *Find the concavity of the integral curve at $x = 1$. Use this to predict if your estimate in Part (a) is too high or too low.*

Solution: The equation $y' = x - y$ implies $y'' = 1 - y'$.

So, when $(x,y) = (1,3)$, we have $y'(1) = 1 - 3 = -2 \quad \Rightarrow y''(1) = 1 - (-2) = 3 > 0$. Since $y''(1) > 0$, the curve is concave up at $(1,3)$. This means the first step is probably an underestimate and we would predict that after 3 steps we still have an underestimate.



Problem 2. *Challenge: Use Euler's method to estimate e.*

–Pick your DE and IC

–Use various numbers of steps to estimate e.

Solution: DE: $y' = y$, IC: $y(0) = 1$. We know this has solution $y = e^x$ which means $y(1) = e$.

So, to estimate e, we will use Euler's method to estimate $y(1)$. The algebra allows us to work out some nice formulas.

In general, a step starts at (x_j, y_j) then $m_j = y_j$ and $m_j h = y_j h$. Thus,

$$x_{j+1} = x_j + h, \quad y_{j+1} = y_j + m_j h = y_j + y_j h = y_j(1+h).$$

This implies,

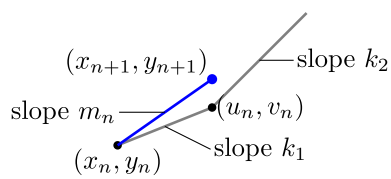
$$y_1 = (1+h)y_0, \quad y_2 = (1+h)y_1 = (1+h)^2 y_0, \quad \ldots \quad y_n = (1+h)^n y_0.$$

Since $y_0 = 1$, we have $y_n = (1+h)^n$. In the following table, n = number of steps to get from $x = 0$ to $x = 1$. We did the computations with a calculator.

$$\begin{array}{llllll}
n = 2 & \Rightarrow h = 1/2, & y(1) & \approx y_2 & = \left(1+\frac{1}{2}\right)^2 & = 2.25 \\
n = 5 & \Rightarrow h = 1/5, & y(1) & \approx y_5 & = \left(1+\frac{1}{5}\right)^5 & = 2.49 \\
n = 10 & \Rightarrow h = 1/10, & y(1) & \approx y_{10} & = \left(1+\frac{1}{10}\right)^{10} & = 2.59 \\
n = 100 & \Rightarrow h = 1/100, & y(1) & \approx y_{100} & = \left(1+\frac{1}{100}\right)^{100} & = 2.705 \\
n = 500 & \Rightarrow h = 1/500, & y(1) & \approx y_{500} & = \left(1+\frac{1}{500}\right)^{500} & = 2.71
\end{array}$$

Problem 3. *Improved Euler is the following algorithm for choosing m_n.*

- *Fixed stepsize: Choose h at the start. It is the same for all steps.*
- *Choice of m_n: (see figure)*



$k_1 = f(x_n, y_n)$; (u_n, v_n) = regular Euler step: $u_n = x_n + h$, $v_n = y_n + k_1 h$.

$k_3 = f(u_n, v_n)$

$$m_n = \frac{k_1 + k_2}{2}$$

- *Then (as always), $x_{n+1} = x_n + h$, $y_{n+1} = y_n + m_n h$.*

Let $y' = y$, $y(0) = 1$. Estimate $y(1)$ using 3 steps and improved Euler.

Solution: Following the algorithm we have

$$h = 1/3, \quad k_1 = y_n, \quad u_n = x_n + h, \quad v_n = y_n + k_1 h, \quad k_2 = v_n, \quad m_n = \frac{k_1 + k_2}{2}$$

We can put the calculation into a table.

n	x_n	y_n	k_1	u_n	v_n	k_2	m_n	$m_n h$
0	0	1	1	0.333	1.333	1.333	1.167	0.389
1	0.333	1.389	1.389	0.667	1.852	1.852	1.620	0.540
2	0.667	1.929	1.929	1.0	2.572	2.572	2.251	0.750
3	1	2.679						

So, $y(1) \approx y_3 \approx 2.679$. The exact answer is $y(1) = e \approx 2.718$. Notice how much faster improved Euler converges than Euler did in the previous problem.