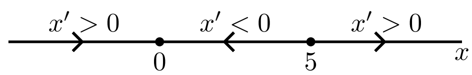
Solutions Day 25, M 3/11/2024
Topic 12: Autonomous systems (day 1 of 2)
Jeremy Orloff

Problem 1. *Consider $x' = 2x(x-5)$. Do a critical point analysis leading to a phase line. Then, in the tx-plane, make a qualitative sketch of some solutions.*

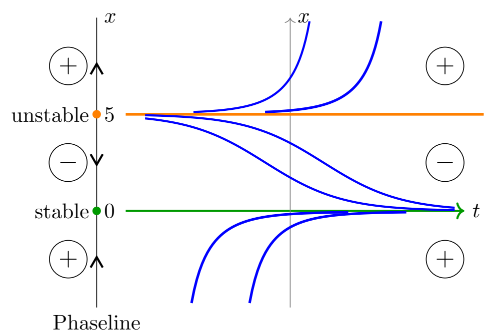
Solution: Critical points: $x' = 2x(x-5) = 0 \quad \Rightarrow x = 0, 5.$

Find intervals where x' is positive or negative:



We've added arrows indicating if a solution $x(t)$ is increasing or decreasing for x in the various ranges.

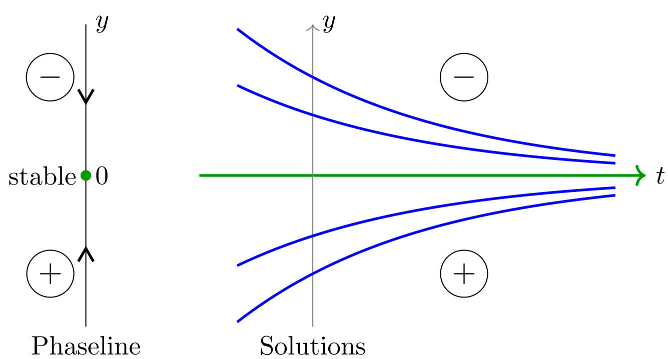
Finally, we draw the phase line vertically and indicate whether the critical points are stable or unstable equilibria. We also put the tx-plane next to the phase line and sketch some integral curves (including the equilibrium solutions).



Problem 2. *For our most important DE: $y' = -y$.*

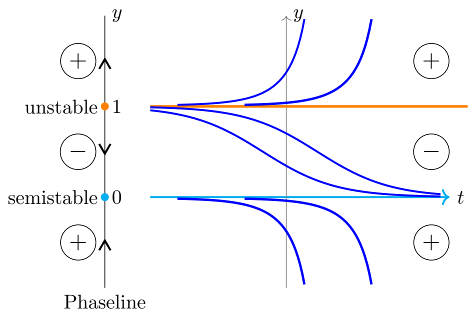
Find the critical points; draw a phase line; classify the equilibria; sketch some solutions in the ty-plane.

Solution: Critical points: $y' = -y = 0 \quad \Rightarrow y = 0$. Here is the phase line and solutions.



Problem 3. *Consider $y' = y^2(y-1)$. Draw the phase line, classify the equilibria, sketch some solutions.*

Solution: Critical points: $y' = y^2(y-1) = 0 \quad \Rightarrow y = 0,\ 1$. Here is the phase line and solutions.

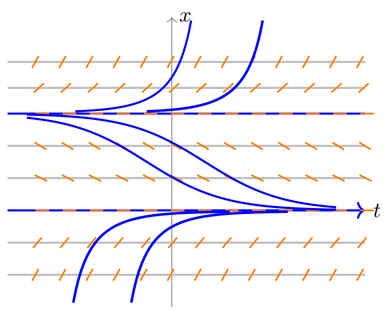


Problem 4. *For the system $y' = y(y-5)$, draw the direction field. Explain how this shows the system is time invariant.*

Solution: Nullcline: $y(y-5) = 0 \quad \Rightarrow y = 0,\ 5$ (horizontal lines)

Isoclines: $y(y-5) = m \quad \Rightarrow y^2 - 5y - m = 0 \quad \Rightarrow y = \dfrac{5 \pm \sqrt{25+m}}{2}$.

We won't bother computing exact values. This shows (as expected) that isoclines are horizontal lines. We'll add a few isoclines for positive m and a few for negative m.



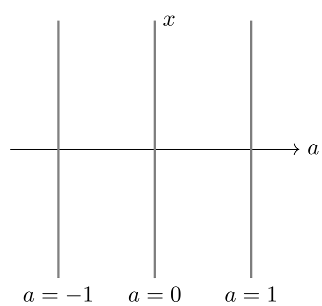
Graphically, time invariance means that the direction field and integral curves are unchanged if we shift the above picture in time, i.e., left or right.

Problem 5. *(This is hinting at bifurcation diagrams.)*

(a) *Find the critical points for $x' = x(a-x)$ (a a constant)*

Solution: Critical points: $x' = x(a-x) = 0 \quad \Rightarrow x = 0,\ x = a$.

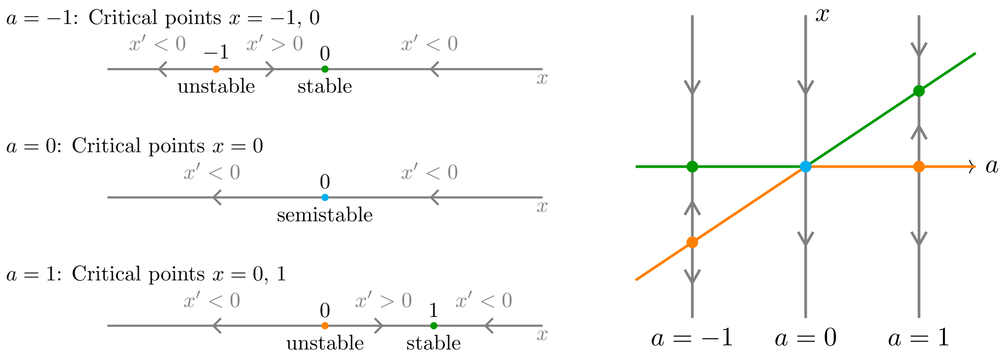
(b) *Copy the ax-plane shown and draw the phase lines for the values of a shown.*



Solution: See figure with solution to Part (c).

(c) *On your ax-plane in Part (b), draw the locus of all points (a, x) where x is a critical point of $x' = x(a - x)$.*

Solution: Critical points $x = 0$ and $x = a$ give two lines in the ax-plane. In the figure below, we've color coded the lines, orange for unstable critical points and green for stable. The stability is also indicated by the arrows around the critical point: arrows point towards the critical point imply it's stable, arrow pointing away imply it's unstable.



3 phase lines and a bifurcation diagram (color coded critical points)