Day 26, T 3/12/2024

Topic 12: Autonomous systems (day 2 of 2)

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1 Agenda

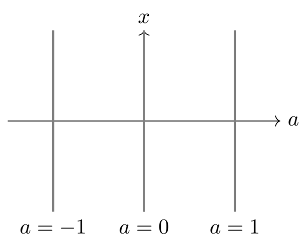
- Warm up problem
- Bifurcation diagrams – autonomous equations with a parameter
- Sustainability = positive stable critical point

2 Warm up problem (#5 from yesterday)

Problem 5. (This is hinting at bifurcation diagrams)

(a) Find the critical points for $y' = y(a - y)$ (a a constant).

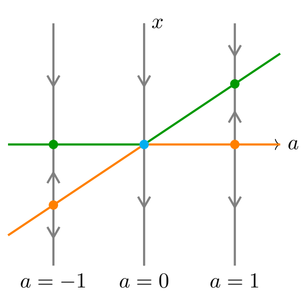
(b) Copy the ay-plane shown and draw the phase lines for the values of a shown.



(c) On your ay-plane, draw the locus of all points (a, y) where y is a critical point of $y' = y(a - y)$.

Solution: **(a)** The critical points are $y = 0$ and $y = a$.

(b) For $a = -1$, 0, 1, we can find the sign of y' on the intervals determined by the critical points and draw the phase lines.



(c) The locus of all critical points consists of the lines $y = 0$ and $y = a$. These are shown in the above diagram. The color coding will be explained below.

Notes

1. The diagram in the previous example is called a bifurcation diagram. The locus of critical points is color coded so that green means the critical point is stable, orange is unstable and cyan is semistable.

2. The system is called sustainable for a given value of a if it is a stable positive critical point. If y is a population, it can last for the long-term at a positive stable equilibrium. In the example above, y is sustainable for all $a > 0$.

3. In the example above, the point $a = 0$ is called a bifurcation point. At bifurcation points, there is a qualitative change in the diagram.

3 Bifurcation diagrams

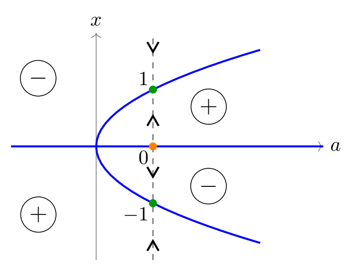
For autonomous equations with a parameter, we can draw a bifurcation diagram.

Goal: Understand how the system behaves for different choices of the parameter.

Example 1. Draw the bifurcation diagram for $y' = ay - y^3$.

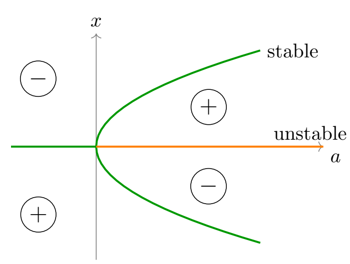
Discuss sustainability and bifurcation points.

Solution: Critical points: $y' = ay - y^3 = 0 \longrightarrow y(a - y^2) = 0 \longrightarrow y = 0$, $y^2 = a$, i.e., a line $y = 0$ and a sideways parabola. The bifurcation diagram is a plot of all the critical points in the ay-plane.



Phase line at $a = 1$

We plotted the critical points. These divide the plane into regions. We mark each region with ⊕ or ⊖ indicating the sign or y'. This allows us to draw (or imagine) the phase line for any fixed value at a. (We added a phase line at $a = 1$.) We can now draw the bifurcation with color coded critical points.



The system is sustainable (has a positive stable equilibrium) for $a > 0$.

$a = 0$ is a bifurcation point. This is the point where there is a qualitative change in the diagram, e.g., for each $a < 0$ there is one critical point and for each $a > 0$ there are three critical points.