Day 29, F 3/15/2024

Topic 13: Linear Algebra

Jeremy Orloff

Topics 13-17: Linear Algebra

In some sense this is just a different terminology to describe things we already know. Here are our main goals.

- Understand the commonalities between linear differential equations and matrices, vectors and systems of linear algebraic equations.
- Learn to express these ideas using the language of linear algebra.
- Learn the basic algebra for working with matrices.
- Use these ideas to solve systems of linear differential equations.

1 Agenda

- Vector spaces
  – Abstract definition
  – All about linearity (superposition), i.e., addition, scalar multiplication, linear combinations
- Matrix multiplication

2 Addition, scalar multiplication, linear combinations

Functions: $x_1(t)$, $x_2(t)$, scalars c_1, c_2

Can add:	$x(t) = x_1(t) + x_2(t)$
Can scale:	$x(t) = c_1 x_1(t)$
Linear combinations:	$x(t) = c_1 x_1(t) + c_2 x_2(t)$

18.02 style vectors: $\mathbf{v_1}$, $\mathbf{v_2}$, scalars c_1, c_2

Can add:	$\mathbf{v} = \mathbf{v_1} + \mathbf{v_2}$
Can scale:	$\mathbf{v} = c_1 \mathbf{v_1}$
Linear combinations:	$\mathbf{v} = c_1 \mathbf{v_1} + c_2 \mathbf{v_2}$

3 Linearity

Both differential operators and matrices can exhibit linearity.

$$\left.\begin{array}{ll} D = \dfrac{d}{dt} & D(c_1x_1 + c_2x_2) = c_1Dx_1 + c_2Dx_2 \\ M = \text{ matrix} & M(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1M\mathbf{v_1} + c_2M\mathbf{v_2} \end{array}\right\} \text{Same idea in both cases.}$$

4 Vector spaces

The word space is used in mathematics to describe a set with extra properities. Math has all kinds of spaces. Here we will be concerned with vector spaces.

The key fact about a vector space is that we can scale and add its elements. First some examples.

Example 1. Let S be the set of functions of t. Clearly, if we add and scale functions, the result is another such function.

Example 2. Let S be the set of all vectors in $\mathbf{R}^2$. Clearly, if we add and scale vectors, the result is another vector.

We say the set S is closed under addition if the sum of two elements from S is another element of S.

Example 3. Let $S =$ the set of all polynomials in x. S is closed under addition. That is, the sum of two polynomials is again a polynomial.

Example 4. Let $S = \{e^{2t} + Ce^{-t}\}$. This is not closed under addition. For example, $(e^{2t} + 3e^{-t}) + (e^{2t} + 4e^{-t}) = 2e^{2t} + 7e^{-t}$ is not in S. (Notice the factor of 2 in front of e^{2t}. It is not allowed for elements of S.)

4.1 Abstract vector spaces

Definition: An abstract vector space is any set where you can add and scale elements and that is closed under those operations. That is, any set where a linear combination of two elements is another element in the set.

The main point of the following two examples is that it is always easy to check if something is a vector space.

Example 5. Show that $S = \{c_1e^{-t} + c_2e^{-7t}\}$ is a vector space.

Solution: These are functions, so we can add and scale them.

Check closed under addition: The sum of two elements of S is again an element of S, i.e.

$$(c_1e^{-t} + c_2e^{-7t}) + (d_1e^{-t} + d_2e^{-7t}) = (c_1 + d_1)e^{-t} + (c_2 + d_2)e^{-7t} \qquad \blacksquare$$

Check closed under scalar multiplication:

$$c(c_1e^{-t} + c_2e^{-7t}) = (c \cdot c_1)e^{-t} + (c \cdot c_2)e^{-7t} \qquad \blacksquare$$

Example 6. Show that the set of solution to the homogeneous linear equation $x'' + 5x' + 6x = 0$ is a vector space.

Solution: We could solve the DE: Solutions $= \{c_1e^{-2t} + c_2e^{-3t}\}$. Showing this is a vector space is exactly like the previous example.

Alternatively, we could work abstractly: The statement that the set of solutions is a vector space is exactly the same as the superposition principle for homogeneous linear DEs. That is, all linear combinations of two solutions are also solutions.

Example 7. (from 18.02) Show that $\mathbf{R}^3 = \{\langle x, y, z\rangle\}$ is a vector space.

Solution: Closed under addition: e.g., $\langle 1,2,3\rangle + \langle 4,5,6\rangle = \langle 5,7,9\rangle$ is in $\mathbf{R}^3$.

Closed under scalar multiplication: e.g., $4\langle 1,2,3\rangle = \langle 4,8,12\rangle$ is in $\mathbf{R}^3$.

5 Matrix multiplication

Example 8. (Reminder from 18.02) Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$. Compute $A\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Solution: $A\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6\cdot 3 + 5\cdot 4 \\ 1\cdot 3 + 2\cdot 4 \end{bmatrix} \begin{matrix} \longleftarrow \text{ 1,1 entry} = \text{Row 1 of } A\cdot \text{ Col 1 of vector} \\ \longleftarrow \text{ 2,1 entry} = \text{Row 2 of } A\cdot \text{ Col 1 of vector} \end{matrix}$

5.1 Key way to view this

$A\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 6 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 3(\text{Col}_1 \text{ of } A) + 4(\text{Col}_2 \text{ of } A) =$ linear combination of the columns of A.

In general, $\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 6 \\ 1 \end{bmatrix} + b\begin{bmatrix} 5 \\ 2 \end{bmatrix} \longleftarrow$ super important!