Solutions Day 29, F 3/15/2024

Topic 13: Linear Algebra

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Problem 1. *Which of the following are vector spaces?*

(a) $\mathbf{R^2}$

(b) *{all functions of t}*

(c) $\{(x, 1) \,|\, x \text{ any value}\}$

(d) *Set of all solutions to* $P(D)x = 0$.

(e) *Set of all solutions to* $P(D)x = \cos(3t)$

(f) $\{(x, 0, z)\}$

Solution: **(a)** $\boxed{\text{Yes}}$. This is a vector space, i.e., it is closed under addition and scalar multiplication. That is, $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$ is in $\mathbf{R}^2$. Likewise, $c\langle x, y\rangle = \langle cx, cy\rangle$ is in $\mathbf{R}^2$.

(b) $\boxed{\text{Yes}}$. If $f(t)$, $g(t)$ are functions and c is a scalar, then $f + g$ and cf are also functions.

(c) $\boxed{\text{No}}$. $(2, 1)$ and $(3, 1)$ are in the set, but $(2, 1) + (3, 1) = (5, 2)$ is not in the set. So the set is not closed under addition.

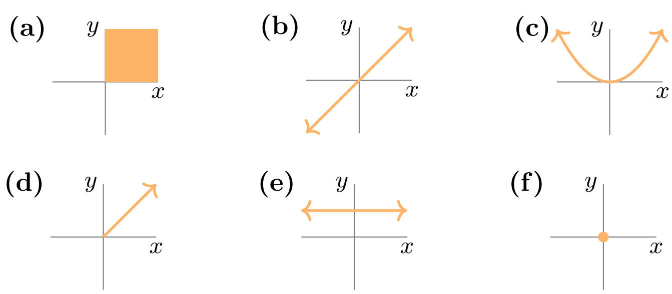
Alternatively: $0 \cdot (2, 1) = (0, 0)$ is not in the set. So it is not closed under scalar multiplication.

(d) $\boxed{\text{Yes}}$. Adding or scaling homogeneous solutions gives another homogeneous solution (this is the superposition principle). So the set is closed under addition and scalar multiplication.

(e) $\boxed{\text{No}}$. Suppose x_p is a solution, i.e., $P(D)x_p = \cos(3t)$. Then $2x_p$ is not a solution, i.e., $P(D)(2x_p) = 2\cos(3t) \neq \cos(3t)$. So $2x_p$ is not in the set of solutions, i.e., the set is not closed under scalar multiplication. (Or $0 \cdot x_p = 0$ is not a solution.)

(f) $\boxed{\text{Yes}}$. Closed under addition and scalar multiplication.

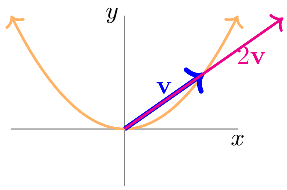
Problem 2. *Are the following shaded (orange) sets vector spaces?*



Solution: **(a)** $\boxed{\text{No}}$. Not closed when scaling by -1.

(b) $\boxed{\text{Yes}}$. Adding two vectors on a line through the origin produces another vector on the same line. Likewise for scaling.

(c) $\boxed{\text{No}}$. Scaling an origin vector with endpoint on the curve produces a vector not on the curve.



(d) $\boxed{\text{No}}$. Not closed when scaling by -1.

(e) $\boxed{\text{No}}$. Scaling any vector by 0 gives $\mathbf{0}$, which is not in the set.

(f) $\boxed{\text{Yes}}$. The set $\{(0,0)\}$ is closed under addition and scalar multiplication.

Problem 3. *Do the following matrix multiplications twice. First as in 18.02. Second as a linear combination of the columns.*

(a) $\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Solution: 18.02 style: $\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4+9 \\ -2+15 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$.

Combination of columns: $\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = -1 \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Solution: 18.02: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 9 + 2\cdot 1 + 3\cdot 0 \\ 4\cdot 0 + 5\cdot 1 + 6\cdot 0 \\ 7\cdot 0 + 8\cdot 1 + 9\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

Combination of columns: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 0 \cdot \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$. (Picks out the middle column.)

Problem 4. *Compute* $\begin{bmatrix} \vec{\mathbf{v_1}} & \vec{\mathbf{v_2}} & \dots & \vec{\mathbf{v_n}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. *(Here, $\vec{\mathbf{v_j}}$ represents the jth column of the matrix.)*

Solution: Linear combination of the columns:

$$\begin{bmatrix} \vec{\mathbf{v_1}} & \vec{\mathbf{v_2}} & \dots & \vec{\mathbf{v_n}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1\vec{\mathbf{v_1}} + c_2\vec{\mathbf{v_2}} + \dots + c_n\vec{\mathbf{v_n}}.$$