Day 31, T 3/19/2024

Topic 14: Row reduction (day 2)

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1 Agenda

- Last time: RREF, pivot and free columns and variables
- Vocabulary: span, independence, basis, rank, dimension
- Column space and null space – meaning and computation
- Connection betweeen A and RREF(A)
- $A\mathbf{x} = \mathbf{b}$: solution = particular + homogeneous
- View: matrix multiplication as a linear transform. (Probably will leave this to the reading and pset.)

2 Row reduction preserves relationships between the columns

Example 1. Let $A = \begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ a_2 & b_2 & a_2+b_2 \\ a_3 & b_3 & a_3+b_3 \end{bmatrix}$

Note: $\text{Col}_3 = \text{Col}_1 + \text{Col}_2$. This is a relation between the columns.

(a) Scale Row_2 by 3 and verify the relation still holds.

(b) Add $3 \cdot \text{Row}_1$ to Row_3 and verify the relation still holds.

Solution: (a) $\xrightarrow{\text{Row}_2 = 3\text{Row}_2} \begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ 3a_2 & 3b_2 & 3(a_2+b_2) \\ a_3 & b_3 & a_3+b_3 \end{bmatrix}$. Clearly we still have $\text{Col}_3 = \text{Col}_1 + \text{Col}_2$.

(b) $\xrightarrow{\text{Row}_3 = \text{Row}_3 + 3\text{Row}_1} \begin{bmatrix} a_1 & b_1 & a_1+b_1 \\ a_2 & b_2 & a_2+b_2 \\ a_3+3a_1 & b_3+3b_1 & a_3+b_3+3(a_1+b_1) \end{bmatrix}$. Clearly, $\text{Col}_3 = \text{Col}_1 + \text{Col}_2$.

3 Vocabulary: span, independence, basis

Example 2. Solve $x'' + 8x' + 7x = 0$. Use linear algebra terms to describe everything.

Solution: Characteristic roots: $r = -1, -7$.

Basic solutions $x_1(t) = e^{-t}$, $x_2(t) = e^{-7t}$,

All solutions = all linear combinations of x_1, x_2: $x(t) = c_1x_1(t) + c_2x_2(t)$.

Linear algebra terminology:

Both $x_1(t)$ and $x_2(t)$ are needed. We call them independent solutions.

$\{x_1, x_2\}$ is a basis of the vector space of solutions.

The vector space of functions $\{c_1x_1(t) + c_2x_2(t)\}$ = span of $\{x_1, x_2\}$.

We'll give more formal definitions of these terms below.

Definition: The span of a set of "vectors" is the set of all their linear combinations. The span is a vector space.

Example 3. (1) If $x_1(t) = e^{-t}$, $x_2(t) = e^{-7t}$, then span $\underbrace{\{x_1, x_2\}}_{\text{2 elements}} = \underbrace{\{c_1e^{-t} + c_2e^{-7t}\}}_{\infty \text{ elements}}$

(2) If $\mathbf{x_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, then span $\underbrace{\{x_1, x_2\}}_{\text{2 elements}} = \underbrace{\left\{c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right\}}_{\infty \text{ elements}}$

4 Null space Null(A)

For a matrix A, Null(A) = the set of all solutions to $A\mathbf{x} = 0$, i.e., the set of homogeneous solutions.

Null(A) is a vector space (= superposition principle = linearity).

Example 4. Find the null space of $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 2 \end{bmatrix}$.

Solution: (Long thorough way)

We have to solve $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Use the augmented matrix $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 1 & 0 & 2 & 0 \end{array}\right]$.

Row reduce to RREF: $\left[\begin{array}{cccc|c} 1 & 0 & -1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 4/3 & 0 \end{array}\right]$ $\longleftarrow$ Note, the augmented zeros never change.

Pivot	Pivot	Free	Free
x_1	x_2	x_3	x_4

Write the free columns in terms of the pivot columns.

$$\text{Col}_3 = -\frac{1}{3}\text{Col}_1 + \frac{2}{3}\text{Col}_2, \qquad \text{Col}_4 = \frac{1}{3}\text{Col}_1 + \frac{4}{3}\text{Col}_2.$$

Rewrite this as $\frac{1}{3}\text{Col}_1 - \frac{2}{3}\text{Col}_2 + \text{Col}_3 = 0, \qquad -\frac{1}{3}\text{Col}_1 - \frac{4}{3}\text{Col}_2 + \text{Col}_4 = 0.$

Since matrix multiplication is a linear combination of the columns we can write this as

$$A \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ i.e., } \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} \text{ are in Null}(A).$$

We have: $\text{Null}(A) = \left\{ c_1 \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

We call $\left\{ \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis of Null(A).

Faster computation. Augmenting by zeros is unnecessary, so we just reduce A to RREF

$$\text{R} = \text{RREF}(A) = \begin{array}{cccc} \left[\begin{array}{cccc} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 2/3 & 4/3 \end{array}\right] \\ \begin{array}{cccc} \text{Pivot} & \text{Pivot} & \text{Free} & \text{Pivot} \\ x_1 & x_2 & x_3 & x_4 \\ 1/3 & -2/3 & 1 & 0 \\ -1/3 & -4/3 & 0 & 1 \end{array} \end{array}$$

We can save space and time by writing the variables underneath the matrix: We set each free variable to 1 and the other to 0. Because the pivot columns are so simple, by thinking of matrix multiplication as a linear combination of the columns, we can find the values of the pivot variables that give a null vector. Finally, we rewrite these as column vectors:

$$\text{Basis of Null}(A) = \underbrace{\left\{ \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{2 elements}}, \qquad \text{Null}(A) = \underbrace{\left\{ c_1 \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\infty \text{ elements}}.$$

Notes: $\text{Null}(A) = \text{Null}(R)$. $\text{Null}(A)$ is a subspace of $\mathbf{R}^4$.

Dimension of $\text{Null}(A) = 2$ = number of free variables = degrees of freedom.

5 Column space Col(A)

Idea: $A\mathbf{x} = \mathbf{b}$ can be solved only if $\mathbf{b}$ is a linear combination of the columns of A. This motivates the following definition.

The column space $\text{Col}(A)$ = the span of the columns of A, i.e., the set of all linear combinations of the columns of A.

Example 5. For $A = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 2 & 4 & 5 & 9 \\ 3 & 6 & 6 & 12 \end{bmatrix}$ find a basis of Col(A).

Solution: By a basis we mean the smallest number of vectors that span the entire space. For Col(A), we don't need the free columns since they are already linear combinations of the pivot columns.

Find RREF(A): $R = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Pivot Free Pivot Free

Use R to identify the pivot columns of A, i.e., Col_1 and Col_3. These are a basis of Col(A). That is,

$$\text{Basis of Col}(A) = \underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}}_{\text{2 elements}}, \qquad \text{Col}(A) = \underbrace{\left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}}_{\infty \text{ elements}}.$$

Notes:

- We needed the pivot columns from A not from R.
- Col(A) is a subspace of $\mathbf{R}^3$ (3 = size of columns = number of rows).
- Null(A) is a subspace of $\mathbf{R}^4$ (4 = number of columns = size of rows).
- Col(A) is not (always) the same as Col(R).
- Dimension of Col(A) = 2 = # pivots = rank.

6 Solving $A\mathbf{x} = \mathbf{b}$

Usual strategy:

Find a particular solution $\mathbf{x_p}$
Find the null space $\mathbf{x_h}$ (general homogeneous solution)
General solution: $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$

- To find $\mathbf{x_p}$, we can set the free variables to 0 and solve the smaller system.
- Can only find a solution if $\mathbf{b} \in \text{Col}(A)$.
- Examples in problems.

7 Vocabulary: independence, basis, dimension, supspace

We have used all of these terms above.

Independence: A set of vectors is independent if no one is a linear combination of the others.

Examples: e^{-t}, e^{-2t}, e^{-3t} are independent.

e^{-t}, e^{-2t}, $e^{-t} + 4e^{-7t}$ are not independent.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are independent

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ are not independent since $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

Basis: A basis for a vector space is a set of independent vectors that span the space. That is, every element in the space is a linear combination of the basis vectors in exactly one way.

Examples: Let S be all solutions to $x'' + 8x' + 7x = 0$, i.e. $S = \{c_1 e^{-t} + c_2 e^{-7t}\}$.

$\{e^{-t}, e^{-7t}\}$ is a basis. $\{e^{-t}, e^{-7t}, 2e^{-t} + e^{-7t}\}$ spans S, but is not a basis.

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^2$.

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is another basis for $\mathbf{R}^2$, i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{(x+y)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(x-y)}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Main idea: Independence = no redundancies – all vectors are need to span the space.

Dimension: The dimension of a vector space is the number of elements in any basis. That is, the number of degrees of freedom.

Examples: $S = \{c_1 e^{-t} + c_2 e^{-7t}\}$ is 2 dimensional.

$\mathbf{R}^2 = \left\{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is 2 dimensional.

Subspace A vector space that is a subset of another vector space is called a subspace.

Example: $\left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a subspace of $\mathbf{R}^2$.