Solutions Day 31, T 3/19/2024

Topic 14: Row reduction (day 2)

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Problem 1. *Let* $A = \begin{bmatrix} 1 & 2 & 2 & 11 \\ 2 & 4 & 1 & 10 \\ 3 & 6 & 0 & 9 \end{bmatrix}$.

We'll tell you that $RREF(A) = R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) *Identify the free and pivot colums of R and A.*

Solution: In both R and A, Col_1, Col_2 are pivot columns and Col_2, Col_4 are free.

(b) *(i) Find a basis for Null(A). Also, find Null(A).*

(ii) How many elements are there in the basis? How many elements are there in Null(A)?

(iii) What is dim(Null(A))?

(iv) Find Null(R).   (Hint: This requires no more work.)

(v) Null(A) is a subspace of $\mathbf{R}^n$*, what is n?*

Solution: Using

$$\begin{array}{cccc} \left[\begin{array}{cccc} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \text{Piv} \;\; \text{Free} \;\; \text{Piv} \;\; \text{Free} \end{array}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & -4 & 1 \end{array}$$

We write the basic null vectors below the matrix as follows:

– We set $\underbrace{x_2 = 1,\, x_4 = 0}_{\text{Free variables can be set freely}}$ and see that $\underbrace{x_1 = -2,\, x_3 = 0}_{\text{Solve for values of pivot variables that make the linear combination of columns 0.}}$

– Likewise, we set $x_2 = 0$, $x_4 = 1$ and see that $x_1 = -3$, $x_3 = -4$.

(i) So a basis of Null(A) is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ and $\text{Null}(A) = \left\{ c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$.

(ii) The basis has 2 elements. Null(A) has infinitely many elements.

(iii) $\dim(\text{Null}(A)) = 2$.

(iv) $\text{Null}(R) = \text{Null}(A)$

(v) Null(A) is a subspace of $\mathbf{R}^4$.

(c) *(i) Find a basis of Col(A). Also, find Col(A).*

(ii) How many elements in the basis? In Col(A)?

(iii) What is dim(Col(A))? What is the rank of A?

(iv) Find Col(R).

(v) Does Col(A) = Col(R)?

Solution: (i) Basis for Col(A) = pivot columns of $A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$\text{Col}(A) = \text{span of basis} = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(ii) The basis has 2 elements. Col(A) has infinitely many elements.

(iii) $\dim(\text{Col}(A)) = 2 = \text{number of pivots} = \text{rank}(A)$.

(iv) $\text{Col}(R) = \text{span of pivot columns of } R = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(v) Clearly $\text{Col}(A) \neq \text{Col}(R)$, e.g., the 3rd entry in vectors in Col(R) is always 0. This is not true of Col(A).

(d) *Find a particular solution to* $A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$. *Do this by setting the free variables to 0 and solving the smaller system by row reduction.*

Solution: The equation to solve is

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 11 \\ 10 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}.$$

Set the free variables, x_2 and x_4 to 0 to get a smaller system

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}.$$

Augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 6 \end{array}\right]$. Row reduction gives $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$.

This corresponds to 3 equations $\begin{cases} x_1 & = 2 \\ x_3 & = -1 \\ 0 & = 0 \end{cases}$.

So we have a particular solution $\boxed{\mathbf{x}_p = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}$.

(e) *Give the general solution to* $A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$.

Solution: From Part (b), $\mathbf{x}_h = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$.

So the general solution is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$.

Problem 2. *Let* $R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) *Give the relations between the free and pivot columns.*

Solution: Pivol columns are C_1, C_3. Free columns are C_2, C_4.

Relations: $C_2 = 3C_1$, $C_4 = 2C_1 + 5C_3$.

(b) *Find a matrix* $A = \begin{bmatrix} 2 & * & 6 & * \\ 5 & * & 1 & * \\ 7 & * & 2 & * \end{bmatrix}$ *that has RREF* R.

Solution: Use the same relations as Part (a) where $C_1 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$, $C_3 = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$.

So, $C_2 = 3C_1 = \begin{bmatrix} 6 \\ 15 \\ 21 \end{bmatrix}$, $C_4 = 2C_1 + 5C_3 = 2 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 34 \\ 15 \\ 24 \end{bmatrix}$.

So, $\boxed{\begin{bmatrix} 2 & 6 & 6 & 34 \\ 5 & 15 & 1 & 15 \\ 7 & 21 & 2 & 24 \end{bmatrix}}$.

Problem 3. *For the matrix* $A = \begin{bmatrix} 1 & 2 & 2 & 11 \\ 2 & 4 & 1 & 10 \\ 3 & 6 & 0 & 9 \end{bmatrix}$, *the pivot columns are Columns 1 and 3.*

Solve $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$.

Solution: As in 1(d), we solve the simpler system $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$.

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 0 & 6 \end{array}\right] \xrightarrow[]{\substack{R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1}} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 1 \\ 0 & -6 & 3 \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

This is 3 equations, the bottom one is $0 \cdot x_1 + 0 \cdot x_3 = 1$. This is impossible, so there are $\boxed{\text{no solutions}}$.