Day 33, R 3/21/2024

Topic 15: Inverse, determinant, transpose

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1 Agenda

- Transpose A^T – swap rows and columns
- Inverses of square matrices
  – diagonal matrices
- Determinants of square matrices
  – diagonal and triangular matrices
- Connections to Null(A)

2 Inverses

A: square $n \times n$ matrix.

A^{-1}: $\quad AA^{-1} = A^{-1}A = I \quad (n \times n$ identity matrix)

($I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad$ etc.)

2×2 inverses: $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (Memorize this.)

- Easy to check that $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (You should do it.)
- If $\underbrace{ad - bc}_{\det A} = 0$ then A^{-1} doesn't exist.

If A^{-1} exists we call A, invertible. If not we call A singular or noninvertible.

2.1 Finding A^{-1}

18.02: Laplace expansion by cofactors (see Topic 15 notes).

Today: Gaussian elimination, i.e., row reduction.

Example 1. Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$. Find A^{-1} using Gaussian elimination.

Solution: Augment A by I: $\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$. Then row reduce until you have I on the left.

$$\xrightarrow{\text{swap rows}} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 6 & 5 & 1 & 0 \end{array}\right] \xrightarrow{\text{Row}_2 = \text{Row}_2 - 6\text{Row}_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -7 & 1 & -6 \end{array}\right] \xrightarrow{\text{Row}_2 = -\frac{1}{7}\text{Row}_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right]$$

$$\xrightarrow{\text{Row}_1 = \text{Row}_1 - 2\text{Row}_2} \left[\begin{array}{cc|cc} 1 & 0 & 2/7 & -5/7 \\ 0 & 1 & -1/7 & 6/7 \end{array}\right]$$

With I on the left side of the augmented matrix, the right side is $A^{-1} = \begin{bmatrix} 2/7 & -5/7 \\ -1/7 & 6/7 \end{bmatrix}$.

Note: If you get free columns on the left, then A^{-1} doesn't exist.

Example 2. $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$. Augmented row reduction gives $\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & * & * & * \\ 0 & -3 & -6 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{array}\right]$

Since we can't get I on the left, there is no A^{-1}, i.e., A is singular.

2.2 Inverses of diagonal matrices are easy

Example 3. Diagonal = all entries off the main diagonal are 0.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}.$$

3 Determinants

A: square $n \times n$ matrix.

18.02: Compute $\det A$ by Laplace expansion (see Topic 15 notes).

For 2×2: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ (Memorize this.)

Can use row reduction to simplify the computation, i.e., get more 0 entries.

- Swap two rows – change sign of determinant
- Scale a row – scale determinant
- Add a multiple of one row to another – no change in determinant

Example 4. Compute $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution: Method 1: Laplace expansion along the top row.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3 + 12 - 9 = 0$$

Method 2: Simplifying first by row reduction.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[\text{det. unchanged}]{\substack{\text{Row}_2 = \text{Row}_2 - 4\text{Row}_1 \\ \text{Row}_3 = \text{Row}_3 - 7\text{Row}_1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow[\text{det. unchanged}]{\text{Row}_3 = \text{Row}_3 - 2\text{Row}_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 0$, the original $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$.

3.1 Determinants of diagonal and triangular matrices are easy

Example 5. The determinant of a diagonal matrix is the product of the diagonal entries.

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 2 \cdot 3 \cdot 4.$$

Example 6. The determinant of a triangular matrix is the product of the diagonal entries.

$$\det \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} = 2 \cdot 5 \cdot 7, \qquad \det \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} = 2 \cdot 4 \cdot 7.$$

4 1803 key point

$$\begin{aligned} \det A = 0 &\longleftrightarrow A^{-1} \text{ doesn't exist } (A \text{ is singular}) \\ &\longleftrightarrow \text{Null}(A) \text{ is nontrivial (not just } \{0\}). \end{aligned}$$

If $\text{Null}(A) = \{0\}$ we say $\text{Null}(A)$ is trivial.